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ANALYSIS OF GENERAL DYNAMIC SYSTEMS INCLUDING ONE NONLINEAR ELEMENT

By

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TECHNICAL REPORT

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OPTIMUM DESIGN OF DELIBERATELY NONLINEAR SERVOMECHANISMS

(NR 374 371)

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ANALYSIS OF GENERAL DYNAMIC SYSTEMS INCLUDING ONE NONLINEAR ELEMENT

Introduction and Summary

In a previous report[1] describing the application of block diagram methods[2-4] to nonlinear circuit problems, it was shown that the mathematical relations for a variety of circuits including one nonlinear element could be indicated by a block diagram of the type given in Fig. 1. In this diagram, $G_1(s)$, $G_2(s)$, and $\beta(s)$ are transfer functions[5] which express in operational form dynamic relations which could also be given as linear differential equations. The forward block represents

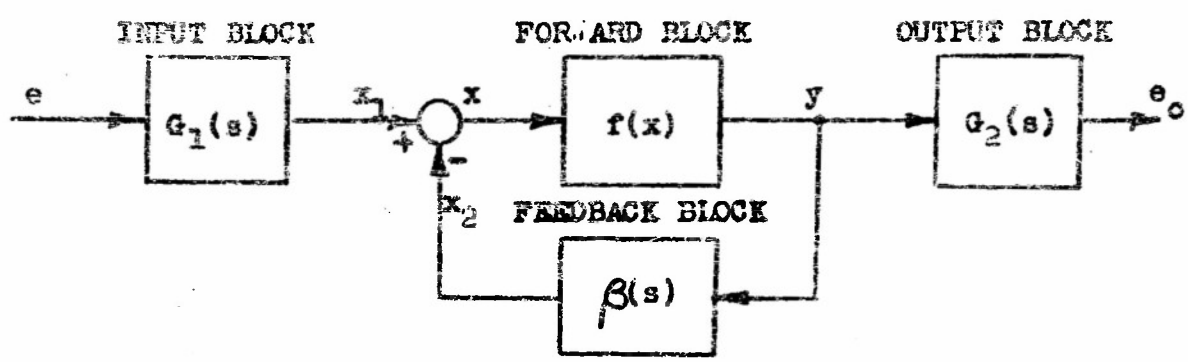


Figure 1. General Block Diagram

a function, f(x), which describes an instantaneous nonlinear relation between the variables x and y. It was also pointed out that circuits exist whose block diagrams cannot be reduced to this general form.

Further study, summarized in this report, indicates that networks of a fairly general class can be described in this way.

A theorem is presented which permits the forward and feedback blocks to be interchanged, if this should prove desirable. The exceptions are likewise found to constitute a general class to which similar methods, with some complications, can be applied.

As indicated previously[1], the variable x_1 may be computed from e by conventional methods; if y can be found, e_o can also be computed as the response of a linear system. The real problem, therefore, is the analysis of feedback system with a nonlinear forward block and a linear feedback block, or vice versa. In this report, a possible step-by-step method for this purpose is described. The method involves the determination of x_1 by operational methods, solution of the feedback system by a combination of an approximation to the superposition integral and a graphical solution of two simultaneous algebraic equations, and finally a calculation of e_o by any appropriate method. A few examples are included to illustrate the method and the question of accuracy is considered briefly.

Although the examples discussed are problems in electric circuits and control systems, the procedures are applicable to other types of dynamic systems by the well-known technique of electro-mechanical analogies or by direct construction of the block diagrams.

Representation of Circuit Elements

The block diagrams for individual linear elements are given in Fig. 2. Two sets of diagrams, called "impedance" and "admittance", are required because of the possibility that either voltages or currents may appear as inputs to the blocks. Mutual inductance and the amplification factor or transconductance of vacuum tubes may also be required in the block diagrams.

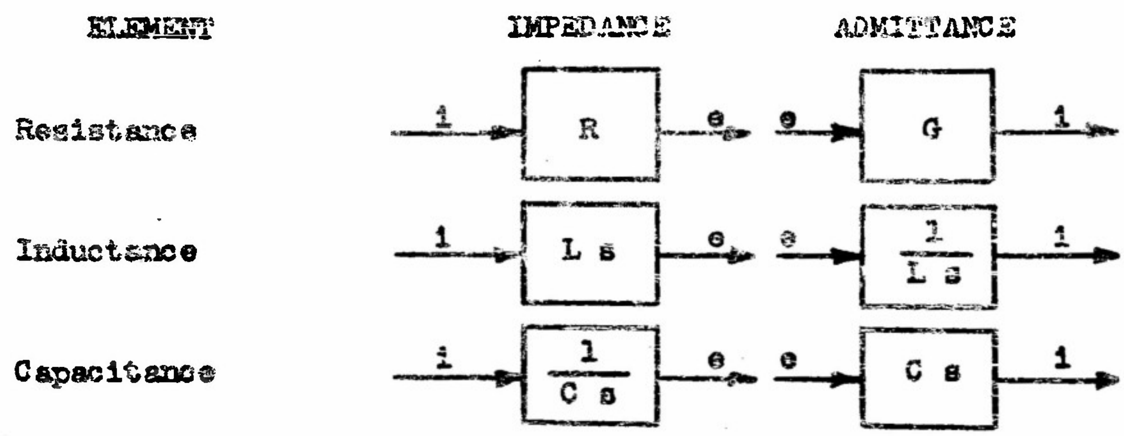


Figure 2. Block Representation of Linear Elements

The block diagram representation of nonlinear elements is not so simple, since it generally requires the specification of a nonlinear function and a dynamic relation. The representations adopted for the common circuit elements are given in Fig. 3. The script symbols ($\mathcal{R}$, $\mathcal{Y}$, Φ, and so forth) denote __functions__, with the output of the block being considered as a function of the input. This function may be given analytically or may only be expressible in graphical or tabular form.

For a nonlinear resistor, the voltage-current relation may be given as

$$e = \mathcal{R}(i) \tag{1}$$

or

$$i = \mathcal{Y}(e), \tag{2}$$

depending on whether the current or voltage is considered to be the independent variable or input.

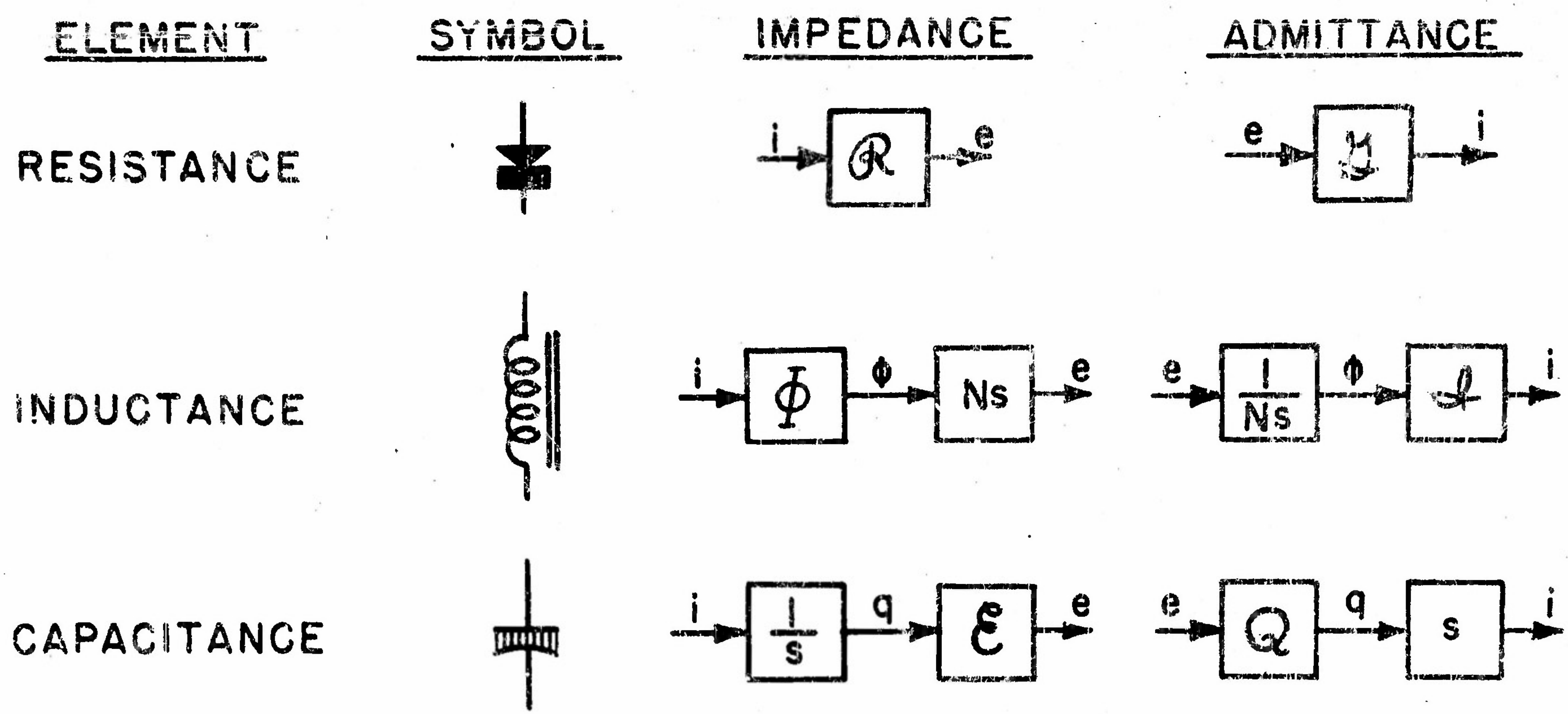


Figure 3. Block Diagram Representation of Nonlinear Elements

A nonlinear inductor may be characterized by two relations:

$$\phi = \Phi(i), \qquad (3)$$

which represents the saturation or magnetization curve, and

$$e = N \frac{d\phi}{dt}, \qquad (4)$$

which is Faraday's law. If several windings appear on the same core, the flux is taken as a function of the net ampere-turns.

A nonlinear capacitor is also described by two relations:

$$q = Q(e), \qquad (5)$$

which is similar to the saturation curve of a magnetic material[6], and

$$i = \frac{dq}{dt}, \qquad (6)$$

a basic definition.

It may be noted in passing that the principle of duality is applicable in nonlinear circuits and that the saturation curves of ferromagnetic and ferroelectric materials correspond closely enough that solutions for one problem can sometimes be used directly as solutions of the associated dual network[6].

General Block Diagram

The block diagram for any network having the configuration shown in Fig. 4(a) or 5(a) can be reduced to the general form given in Fig. 1. In these networks, the nonlinear element appears between two four-terminal networks, in either a series or shunt position. Although nonlinear resistors are shown, any of the other nonlinear elements may be substituted.

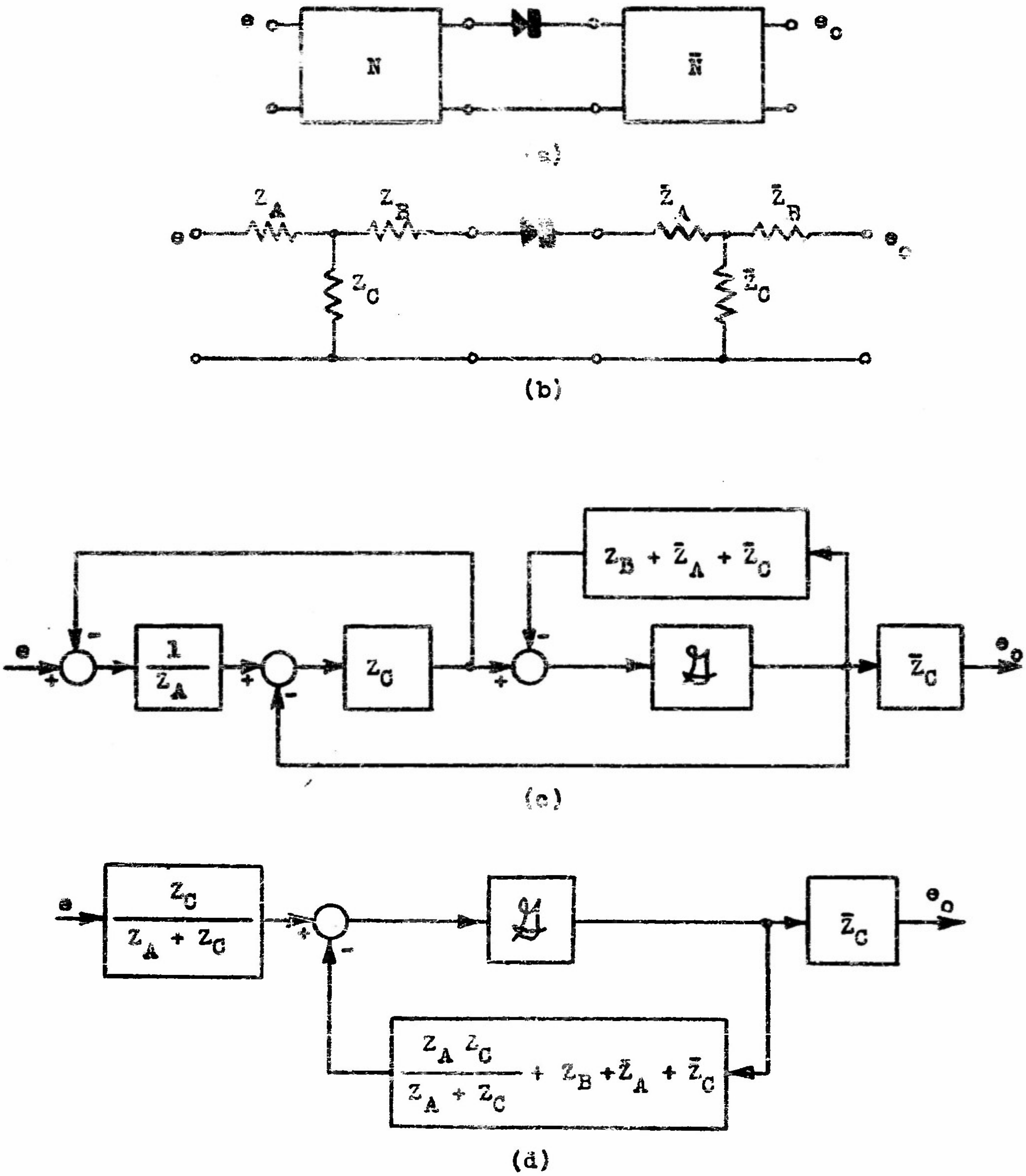


Figure 4. Block Diagram for Series Nonlinear Element

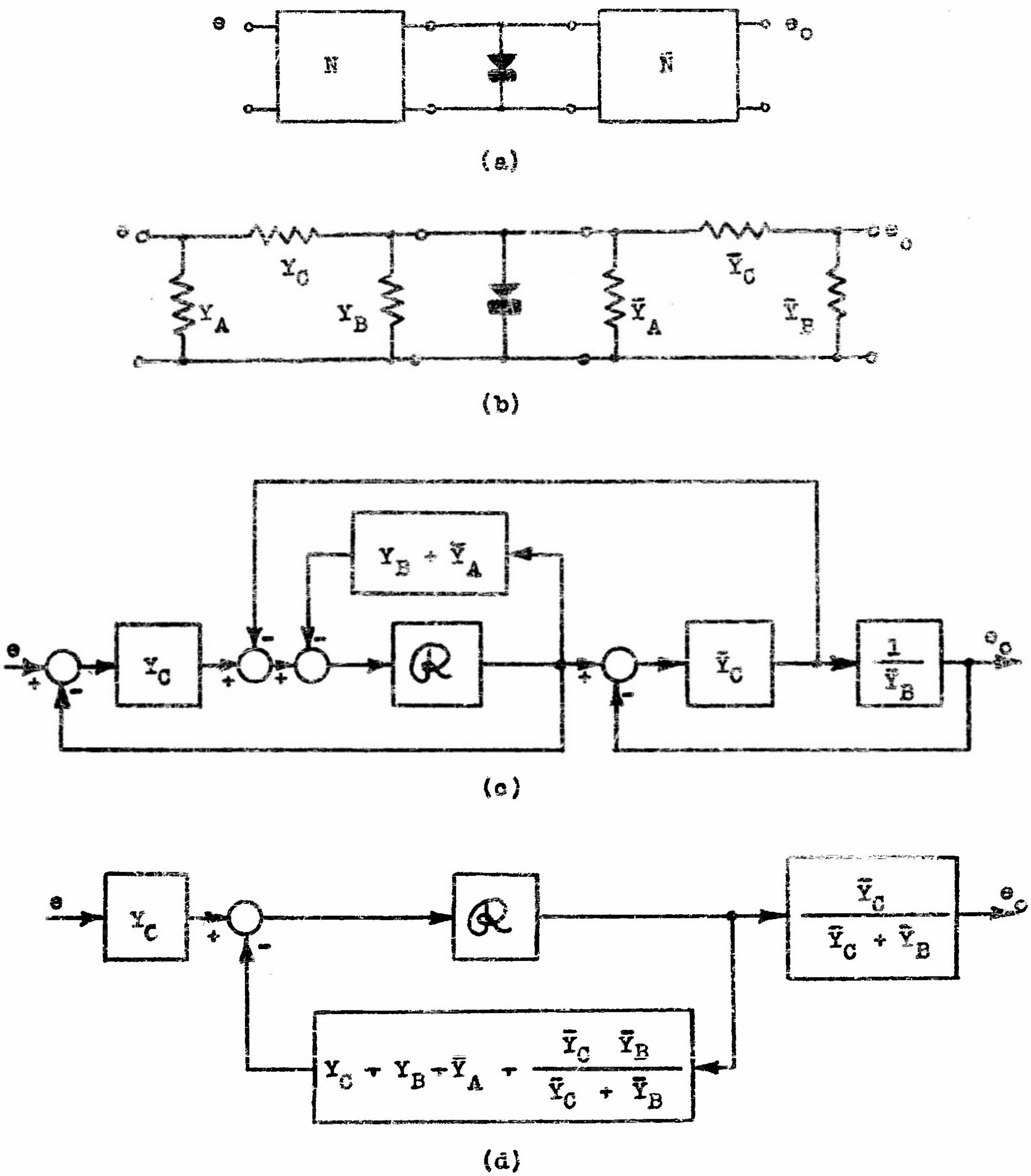


Figure 5. Block Diagram for Shunt Nonlinear Element

The four-terminal networks may be characterized by their equivalent T- or π-sections, whichever is most convenient, as shown in Figs. 4(b) and 5(b). The impedances of the generator supplying the network and the measuring device at the output are assumed to be included in the networks N and $\bar{N}$. Should it be desirable, the networks can also be specified by their open-circuit impedance parameters or the short-circuit admittance parameters[7].

The original block diagrams, based on the circuit equations, are given in Figs. 4(c) and 5(c). The final diagrams, obtained by a few simple block transformations[1-4], are given in Figs. 4(d) and 5(d). Study of these final diagrams indicates that they can also be obtained by direct application of standard network theorems; Fig. 4(d) can be derived from the circuit diagram by the use of Thevenin's theorem, and Fig. 5(d) by Norton's theorem.

It will be observed that a series nonlinear element is best described by its admittance function (voltage input and current output), and that a shunt element is described by an impedance relation. Likewise, it may be seen that $\bar{Z}_B$ does not appear in Fig. 4(d) and Y_A does not appear in Fig. 5(d).

The same type of final block diagram is obtained if the four-terminal networks are coupled by a nonlinear (iron-core) transformer, as suggested by Fig. 6 of the earlier report[1].

Exceptions

If the series element is shunted or bridged by an unlike linear element, or if an unlike linear element is placed in series with the shunt element, a network is produced whose block diagram does not reduce to the general form of Fig. 1. Such networks are shown in Figs. 6 and 7.

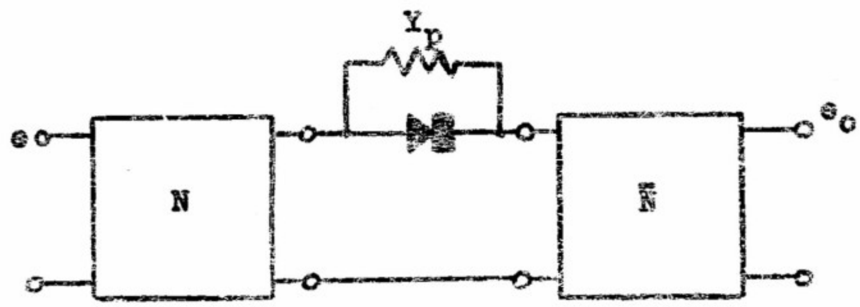


Figure 6. Parallel Combination between Two Networks

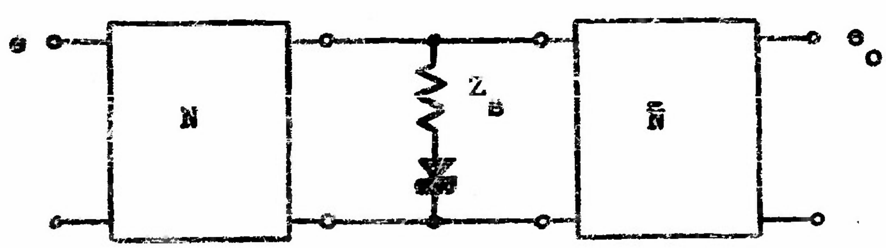


Figure 7. Series Combination between Two Networks

A possible method for dealing with exceptions of this type is suggested in Fig. 8. A circuit diagram for the network of Fig. 6, using T-section representations for the four-terminal networks and expressing the elements in terms of the impedance parameters, is given in Fig. 8(a). In developing the block diagram given in Fig. 8(b), Thevenin's theorem is applied to the left-hand network (N) to obtain an open-circuit voltage, $e(Z_{12}/Z_{11})$, and an equivalent source impedance, $1/Y_{22}$. The voltage across the parallel combination of Y_p and the nonlinear element is then computed by subtracting the voltages across $(1/Y_{22})$ and $\bar{Z}_{11}$ from the open-circuit voltage. With the voltage across the parallel combination available, the current entering the right-hand network ($\bar{N}$) can be computed and then used to find the output voltage, as well as the internal voltage drops.

For convenience, the feedback block in Fig. 8(b) is called "β" in developing the final diagram given in Fig. 8(c). This diagram can be obtained by block transformations or by applying Norton's theorem to the equivalent circuit from which Fig. 8(b) was derived. The final diagram bears some resemblance to Fig. 1, although the nonlinear block now appears in the feedback path, and can be handled by similar methods.

The network shown in Fig. 7 may be treated in essentially the same way and the details are not included here.

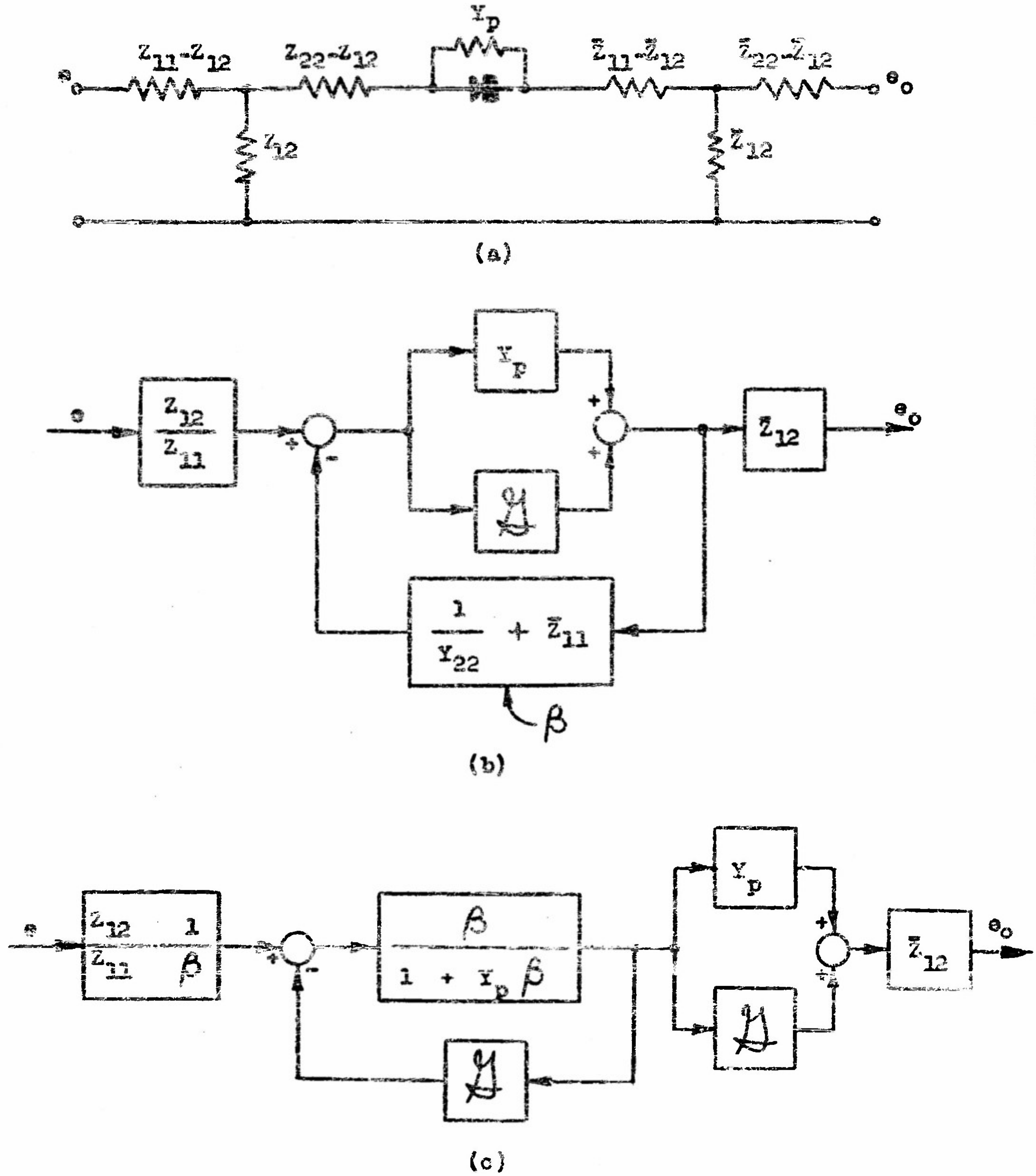


Figure 8. Block Diagram for Network of Figure 6

A Theorem

The block transformation shown in Fig. 9(a), although not widely known or used, can easily be checked by solving the basic equations of the feedback systems or by application of the usual relation

$$A = \frac{A_1}{1 + A_1 A_2} = \frac{\frac{1}{A_2}}{1 + \frac{1}{A_1 A_2}} \tag{7}$$

If the forward block is nonlinear, the transformation shown in Fig. 9(b) is applicable; this follows from the fact that the equations may be written as

$$x = x_1 - x_2 \tag{8}$$

$$y = f(x) \tag{9}$$

$$x_2 = \beta(s) \; y \tag{10}$$

or as

$$x_2 = x_1 - x \tag{11}$$

$$y = \frac{1}{\beta(s)} x_2 \tag{12}$$

$$x = g(y) \tag{13}$$

where the function g is related to f in such a way that

$$x = g\left[f(x)\right] = x \tag{14}$$

It may be said that the forward and feedback blocks can be interchanged by the substitution of their inverses or reciprocals. Application of this theorem permits the feedback loop in Fig. 8(c) to be reduced to the standard form.

An illustration of the theorem is presented in Fig. 10, where a specific circuit problem is considered.

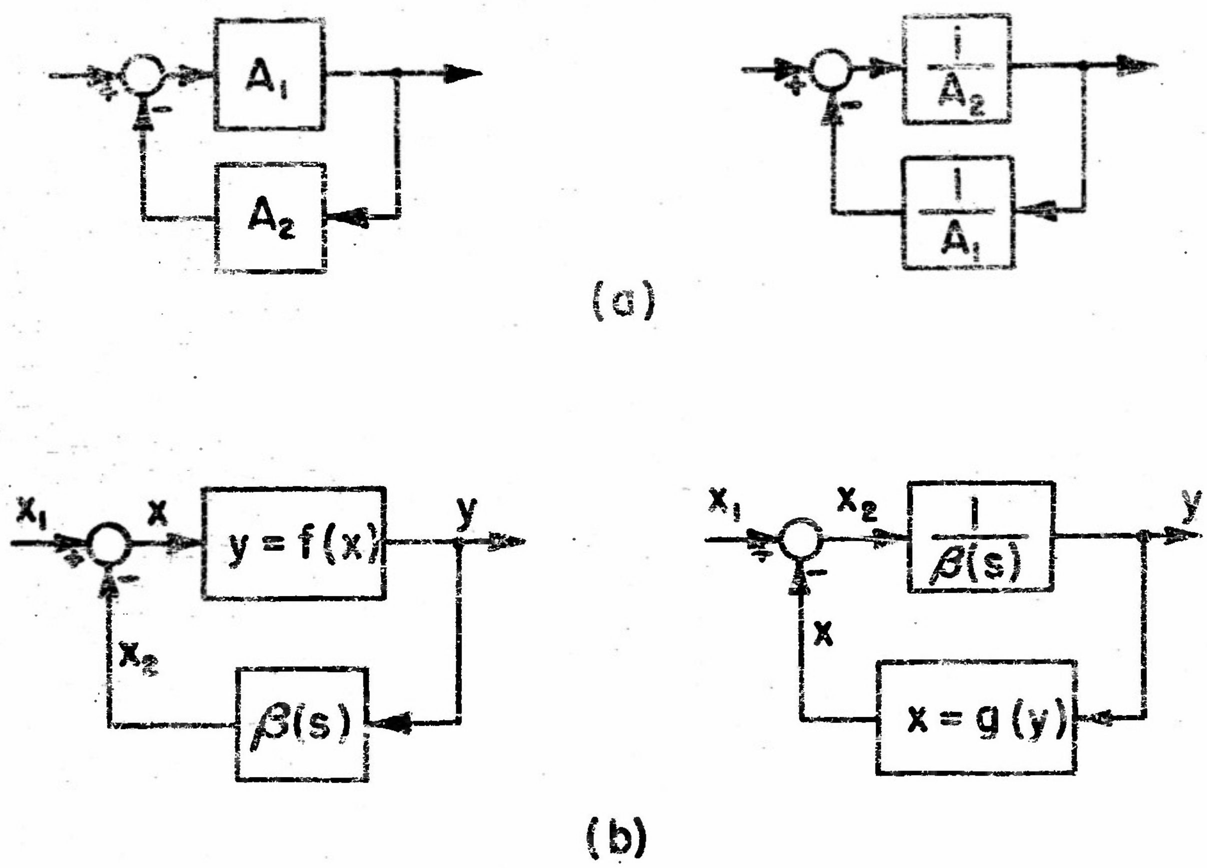


Figure 3. A Block Diagram Transformation

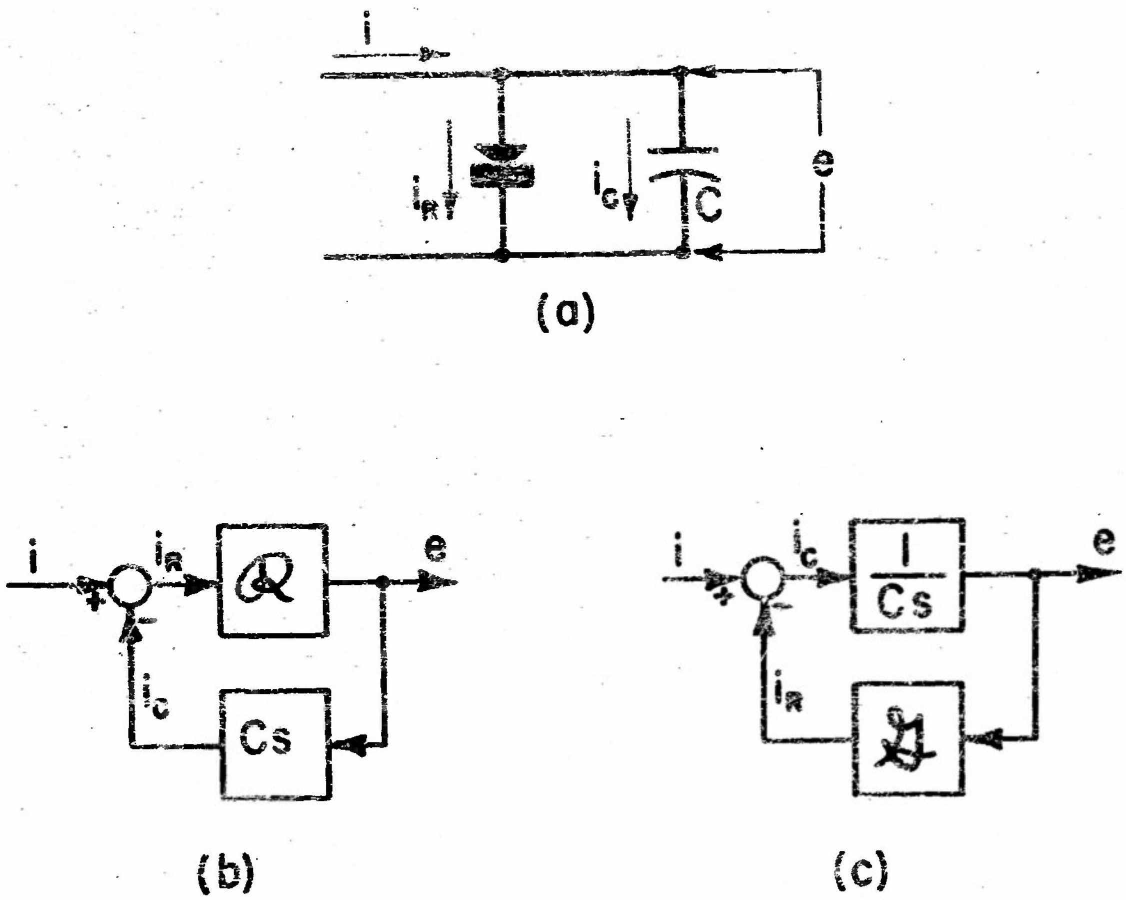


Figure 10. Circuit Illustration of Block Diagram Transformation

General Method of Solution

The analytical problem posed by Fig. 1 may be stated as follows: given a complete description of the linear and non-linear portions of the system and a specified input, $e(t)$, what is the corresponding output, $e_o(t)$? A general approach to this problem is described and illustrated in the remainder of the report.

The first step in the solution is the determination of $x_1(t)$, which may be accomplished by any of several methods. Since the input block is linear, the differential equation relating $e(t)$ and $x_1(t)$ can be recovered and solved by classical methods. Using operational methods, we would have

$$x_1(t) = \mathcal{L}^{-1}\left[G_1(s) \quad E(s)\right] \qquad (15)$$

where $E(s)$ is the transform of $e(t)$. If the steady-state solution for a periodic input is required, the transfer function $G_1(s)$ can be converted into an expression for magnitude and phase at real frequencies by the substitution of $j\omega$ for s. In the general case, where neither Laplace nor Fourier transforms are especially convenient, it is possible to write[8]

$$x_1(t) = \int_0^t e(t - \tau)\, g_1(\tau)\, d\tau \qquad (16)$$

where $g_1(t) = \mathcal{L}^{-1}\left[G_1(s)\right]$, the impulse response or weighting function for the input block, and to determine $x_1(t)$ approximately to any desired accuracy.

The real problem is the calculation of the response of the feedback system, that is, the determination of $y(t)$ for a given $x_1(t)$. Using Eqs. (8)-(10), this problem may be written in the time domain as an integral equation

$$x(t) = x_1(t) - \int_0^t \beta(\tau) \; f\left[x(t-\tau)\right] d\tau \qquad (17)$$

or

$$x(t) = x_1(t) - \int_0^t \beta(t-\tau) \; f\left[x(\tau)\right] d\tau \qquad (18)$$

where $\beta(t) = \mathcal{L}^{-1}\left[\beta(s)\right]$, the impulse response or weighting function of the feedback block. Equations (17) and (18) are both special cases of Lalesco's nonlinear integral equation[9], which has been applied to circuit problems by Keller[10] and Pipes[11]. In the analytic treatment of this type of equation, a sequence of functions is obtained whose limit is the solution. The main difficulty with this method is the complexity of the integrands encountered. The necessity for an analytic description of the nonlinearity may also be troublesome.

Approximate solutions of the system equations can be obtained by step-by-step methods, using either one of two procedures based on Eqs. (8)-(10), which will probably be simpler to apply than the iteration method just described. In the first method, the feedback block is handled by writing Eq. (10) as

$$x_2(t) = \int_0^t \beta(\tau) \; y(t-\tau) \, d\tau \; ; \qquad (19)$$

this integral is then evaluated numerically by any of the well-known methods for approximate integration[12-15]. This approach has been used by Mazelsky and Diederich[16] in work on strictly linear systems.

In the second method, $y(t)$ is resolved into component signals (step functions, rectangular or triangular pulses, and so forth) for which the response of the feedback block can be computed; the variable $x_2(t)$ is obtained by superposition of the responses to the separate components of $y(t)$. This method has been applied to nonlinear systems by Tustin[17-18], Madwed[19], and Harmer[20]; some of the mathematical questions involved have been discussed by Brown[21]. In these applications, the sequence of numbers representing the response of the linear portions of the system to the basic input component has been found by substituting difference operators for differential operators, rather than by direct calculation.

The first method will be employed in this report, since it appears to offer some advantages. Unlike the other methods, no attempt is made to approximate the variables in components or to express the operation of differentiation in terms of differences. Errors peculiar to the first method are made in the approximate integration, with the result that their size can readily be estimated. With a knowledge of the weighting function (calculated in advance by ordinary operational methods) and the input x_1, a suitable time increment can be selected. The arithmetic operations are routine and a standard type of graphical computation is used.

Having obtained an approximate solution for $y(t)$, it is still necessary to find $e_o(t)$. In some cases, as illustrated by the examples, $e_o(t)$ and $x_2(t)$ are simply related; in other cases, a second approximate integration may be required.

Details of Solution

Before presenting any examples, a more detailed description will be given of the method used to obtain the response of the feedback system.

The impulse response of the feedback block is calculated by taking

$$\beta(t) = \mathcal{L}^{-1}\left[\beta(s)\right] \tag{20}$$

Since $\beta(s)$ is normally the ratio of two polynomials in s, the relative orders of the numerator and denominator are of interest. If $\beta(s)$ is, for example, of the form

$$\beta(s) = \frac{3}{s^2 + 4s + 13} \tag{21}$$

the impulse response is found to be

$$\beta(t) = e^{-2t} \sin 3t \tag{22}$$

from any transform table[8].

If the numerator and denominator are of the same order, as in the case

$$\beta(s) = K \frac{s}{s+1} \tag{23}$$

it is advisable to write (by division)

$$\beta(s) = K - K \frac{1}{s+1} \tag{24}$$

The impulse response is then

$$\beta(t) = K\,\delta(0) - K e^{-t} \tag{25}$$

where $\delta(0)$ is a unit impulse at zero time.

Such impulsive components actually simplify the problem, since they imply a strong dependence of the present value of $x_2(t)$ on the present value of $y(t)$. In fact, if $K\,\delta(0)$ were the only term in $\beta(t)$, we would have

$$x_2(t) = K \quad y(t) \tag{26}$$

The feedback block would therefore be represented by a linear relation between the instantaneous values of x_2 and y, and could be combined with the nonlinear forward block by algebraic means.

If the order of the numerator of $\beta(s)$ exceeds the order of the denominator, the theorem given on p. 12 may be invoked. The reciprocal of $\beta(s)$ is then placed in the forward block and the nonlinear relation is inverted and put in the feedback block. This possibility is illustrated in one of the examples.

In the usual case, then, a time interval h will be chosen and the values of $\beta(t)$ computed for $t = k\,h$, where $k = 0, 1, 2, 3, \cdots$. This sequence will be written

$$\beta_0, \ \beta_1, \ \beta_2, \ \beta_3, \ \cdots \tag{27}$$

where the subscripts correspond to the various values of k. Similarly, the values of $y(t)$ will be written as the sequence

$$y_0, \quad y_1, \quad y_2, \quad y_3, \ \cdots \tag{28}$$

To determine the values of $x_2(t)$ at the times $t = n\,h$, the integral of Eq. (19) will be evaluated by the trapezoidal rule. More elaborate and accurate methods are available[12-15] and have been used in similar computations[16], but the choice of the trapezoidal rule results in a particularly convenient and simple procedure. According to this rule, the integral is

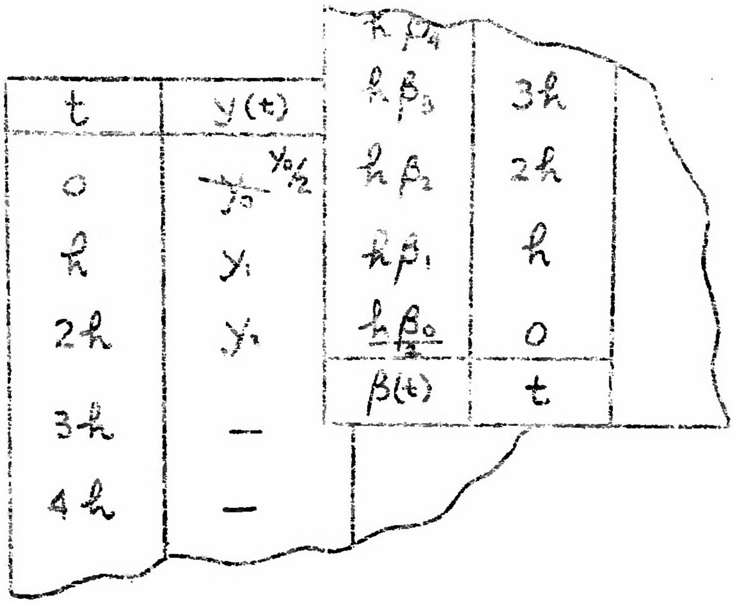
$$x_2(nh) \approx h\left[\frac{1}{2}\beta_0 y_n + \beta_1 y_{n-1} + \beta_2 y_{n-2} + \cdots + \beta_{n-1} y_1 + \frac{1}{2}\beta_n y_0\right] \tag{29}$$

[If $\beta(t)$ contains an impulsive component of magnitude K, Ky_n should be added to Eq. (29).] Since $h\beta_k$ always appears in combination, it is convenient to absorb h into the β-sequence given in (27). The first member of the β-sequence can also absorb the factor 1/2, as can the first member of the y-sequence. The two sequences are then

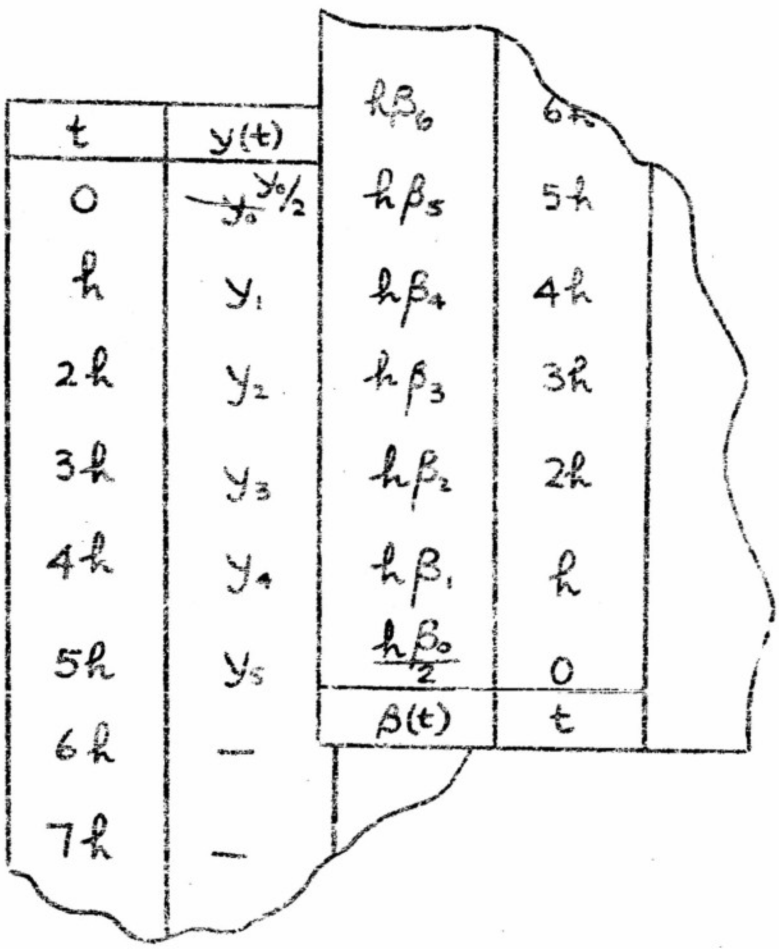
$$\frac{h\beta_0}{2},\ h\beta_1,\ h\beta_2,\ h\beta_3,\ \ldots\ldots \tag{30}$$

$$\frac{y_0}{2},\ y_1,\ y_2,\ y_3,\ \ldots \tag{31}$$

The process of multiplications and additions indicated by Eq. (29) is carried out conveniently on a desk calculator. The sequence (30) is computed in advance and arranged in tabular form along the left-hand edge of a piece of paper, starting at the bottom. The values of y(nh) are determined during the course of the computation and are recorded in a vertical column on another piece of paper. The technique used to keep track of the operations in Eq. (29) is shown in Fig. 11.



$$
\begin{aligned}
& \frac{y_0}{2}\, h\beta_2 \\
+\; & y_1\, h\beta_1 \\
+\; & y_2\, \frac{h\beta_0}{2} \\
\hline
=\; & x_2(2h)
\end{aligned}
$$



$$
\begin{aligned}
& \frac{y_0}{2}\, h\beta_5 \\
+\; & y_1\, h\beta_4 \\
+\; & y_2\, h\beta_3 \\
+\; & y_3\, h\beta_2 \\
+\; & y_4\, h\beta_1 \\
+\; & y_5\, \frac{h\beta_0}{2} \\
\hline
=\; & x_2(5h)
\end{aligned}
$$

Figure 11. Portion of Table Showing Method Used to Find $x_2(nh)$

It will be noted that, unless $\beta_0 = 0$, $x_2(nh)$ will depend on y_n, as well as the previous values of y. However, y_n is also dependent on $x_2(nh)$ in another way, as indicated by the relations

$$y_n = f\left[x(nh)\right] \tag{32}$$

$$x(nh) = x_1(nh) - x_2(nh) \tag{33}$$

from Eqs. (8) and (9). If we denote by $S(nh)$ all terms of Eq. (29) except the first, we may write

$$x_2(nh) = r\,y_n + S(nh) \tag{34}$$

where $r = h\,\beta_0/2$. Equation (33) may therefore be written

$$x(nh) = x_1(nh) - r\,y_n - S(nh) \tag{35}$$

$$= \bar{S}(nh) - r\,y_n \tag{36}$$

where $$\bar{S}(nh) = x_1(nh) - S(nh). \tag{37}$$

Since $x_1(t)$ has already been calculated for all values of t and $S(nh)$ depends only on past values of y, the present values of x and y can be found by a simultaneous solution of Eqs. (32) and (36).

The simultaneous determination of $x(nh)$ and $y(nh)$ can be made very conveniently by a graphical method shown in Fig. 12. This process is exactly the same as the one used for the large-signal analysis of vacuum tubes having resistive loads. The corresponding quantities are:

$y \longleftrightarrow i_b \qquad\qquad \bar{S} \longleftrightarrow E_{bb}$

$x \longleftrightarrow e_b \qquad\qquad r \longleftrightarrow R_L$

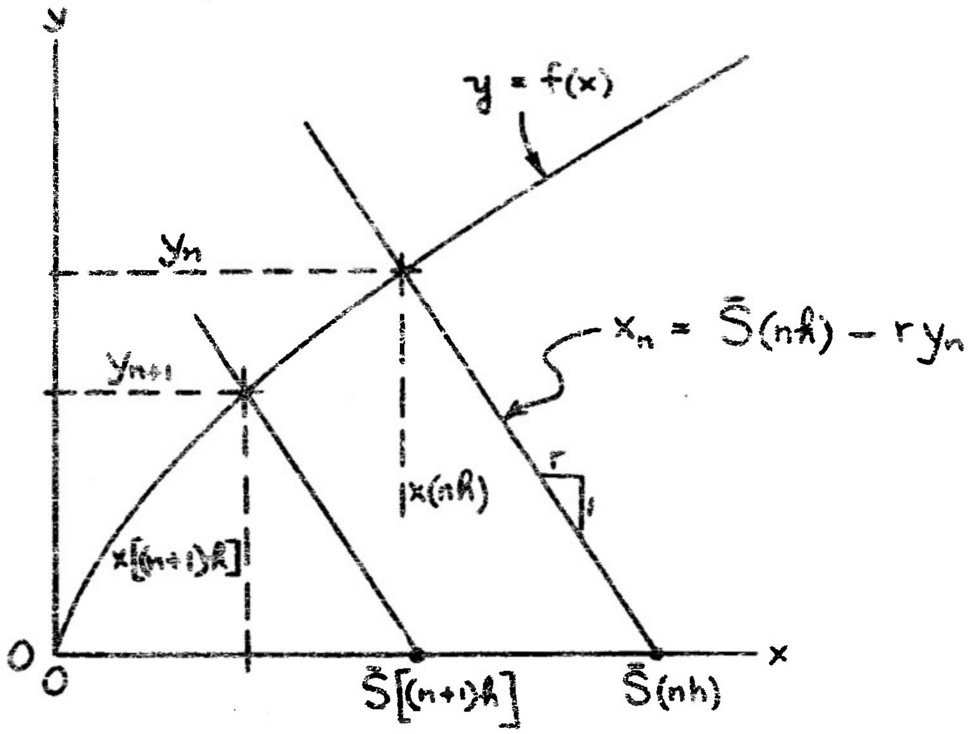


Figure 12. Graphical Solution for y_n and x(nh)

Inasmuch as the slope of the straight line given by Eq. (36) is a constant and only $\bar{S}$ varies, a single straight line of the proper slope can be drawn on a piece of semi-transparent tracing paper which may be superimposed on the plot of y as a function of x. As soon as $\bar{S}(nh)$ has been computed, the tracing paper is placed in the correct position, the values of y_n and x(nh) are found at the intersection of the curves and entered in the table of values, from which $\bar{S}\left[(n+1)h\right]$ is computed. The cycle is then repeated as many times as necessary.

Examples

(1) Series Resonant Circuit with Nonlinear Inductor

A series circuit consisting of a resistor, capacitor, and nonlinear inductor is shown in Fig. 13(a). The problem is to determine the current caused by a 5-volt step input.

The original block diagram, given in Fig. 13(b), is reduced to the standard form given in Fig. 13(c) by moving the block containing (1/s) to the left of the summing point and into the feedback path. Since

$$E(s) = \frac{5}{s} \tag{38}$$

and

$$G_1(s) = \frac{1}{s} \; , \tag{39}$$

we have

$$X_1(s) = \frac{5}{s^2} \tag{40}$$

and therefore

$$x_1(t) = 5t \; . \tag{41}$$

The feedback function is

$$\beta(s) = \frac{2}{s} + \frac{2}{s^2} \tag{42}$$

giving

$$\beta(t) = 2 + 2t \tag{43}$$

For $h = 0.2$, the modified β-sequence is therefore

$$\frac{0.40}{2}\, ,\ 0.48,\ 0.56,\ 0.64,\ 0.72,\ 0.80,\ 0.88,\ \cdots \tag{44}$$

The quantity r is therefore 0.20, and the general equation for x_n is

$$x_n = \bar{S}(nh) - 0.20\, y_n \tag{45}$$

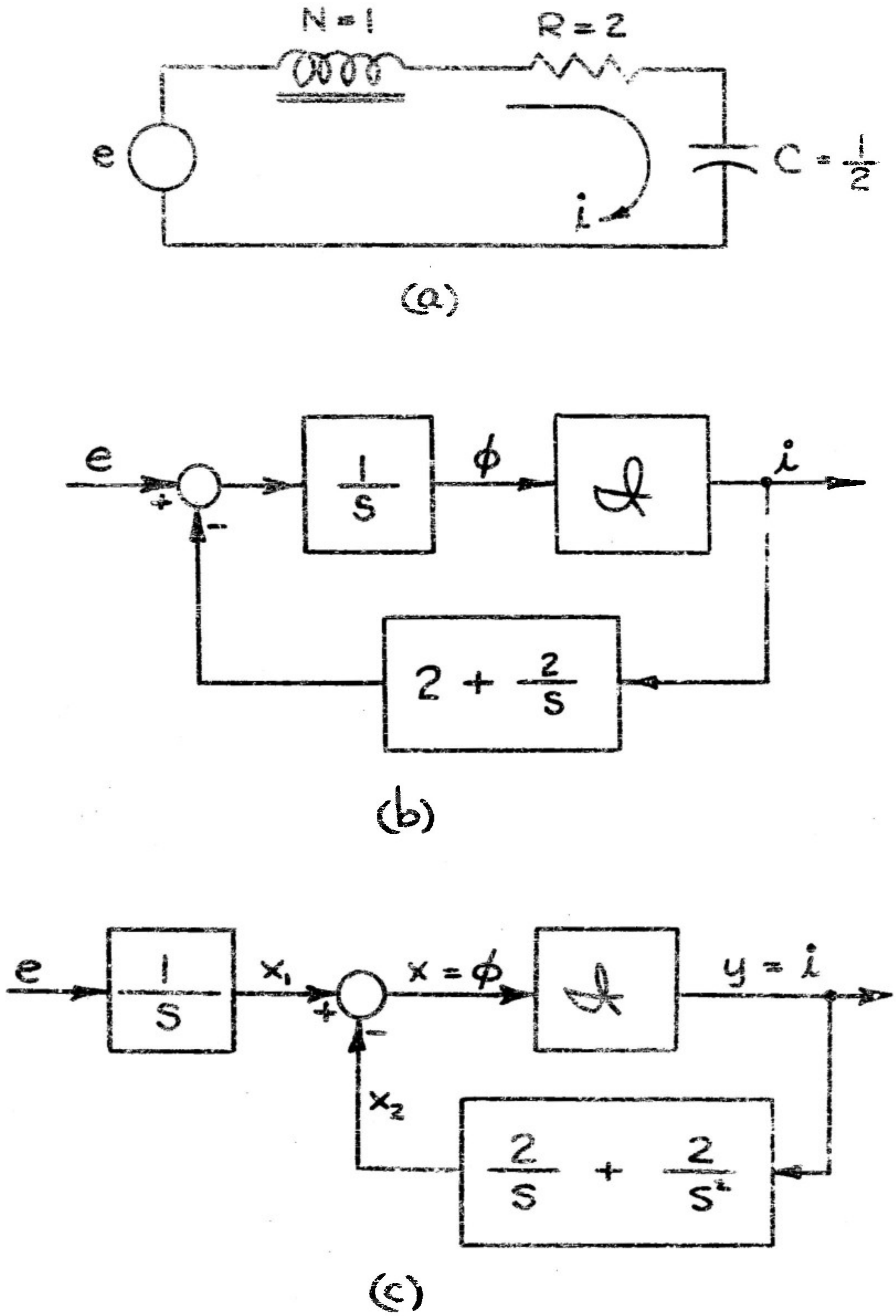


Figure 13. Series Resonant Circuit and Block Diagram

To check the accuracy of the method, a linear case is treated first. The relation between ϕ and i is assumed to be

$$\phi = i$$

or

$$x = y \qquad (46)$$

The initial conditions are taken to be $q(0) = 0$, $i(0) = 0$.

After laying out the table shown on the next page, values of x_1 are computed from Eq. (41) and entered in the x_1-column. The initial value of y, $y_0 = 0$, is then entered at the top of the y-column. The next entries are computed as follows:

From Eqs. (29) and (44),

$$\begin{aligned} S(0.2) &= \beta_1 y_0 \\ &= (0.48)(0) \\ &= 0 \end{aligned} \qquad (47)$$

From Eqs. (37) and (41),

$$\begin{aligned} \bar{S}(0.2) &= x_1(0.2) - S(0.2) \\ &= 1.000 \end{aligned} \qquad (48)$$

Substituting the required numbers and solving Eqs. (45) and (46) simultaneously, we get

$$x(0.2) = y(0.2) = 0.833 \qquad (49)$$

The graphical construction, shown in Fig. 14, is unnecessary in this problem, since Eqs. (45) and (46) can be solved analytically to give

$$y_n = \frac{\bar{S}(nh)}{1.20} \qquad (50)$$

t	x_1	s	$\bar{s}$	x_2	$x = \phi$	$y = i$
0	0	-	-	0	0	0
0.2	1	0	1.000	0.167	0.833	0.833
0.4	2	0.400	1.600	0.667	1.333	1.333
0.6	3	1.107	1.893	1.422	1.578	1.578
0.8	4	2.038	1.962	2.365	1.635	1.635
1.0	5	3.122	1.878	3.435	1.565	1.565
1.2	6	4.304	1.696	4.586	1.414	1.414
1.4	7	5.538	1.462	5.782	1.218	1.218
1.6	8	6.792	1.208	6.993	1.007	1.007
1.8	9	8.041	0.959	8.201	0.799	0.799
2.0	10	9.259	0.741	9.382	0.618	0.618

Table I. Calculations for Series Resonant Circuit with Linear Inductor

At the next step, we have

$$\begin{aligned} s(0.4) &= \beta_1 y_1 + \beta_2 y_0 \\ &= (0.48)(0.833) + (0.56)(0) \\ &= 0.400 \end{aligned} \qquad (51)$$

$$\begin{aligned} \bar{s}(0.4) &= x_1(0.4) - s(0.4) \\ &= 2.000 - 0.400 \\ &= 1.600 \end{aligned} \qquad (52)$$

$$x(0.4) = y(0.4) = 1.333 \qquad (53)$$

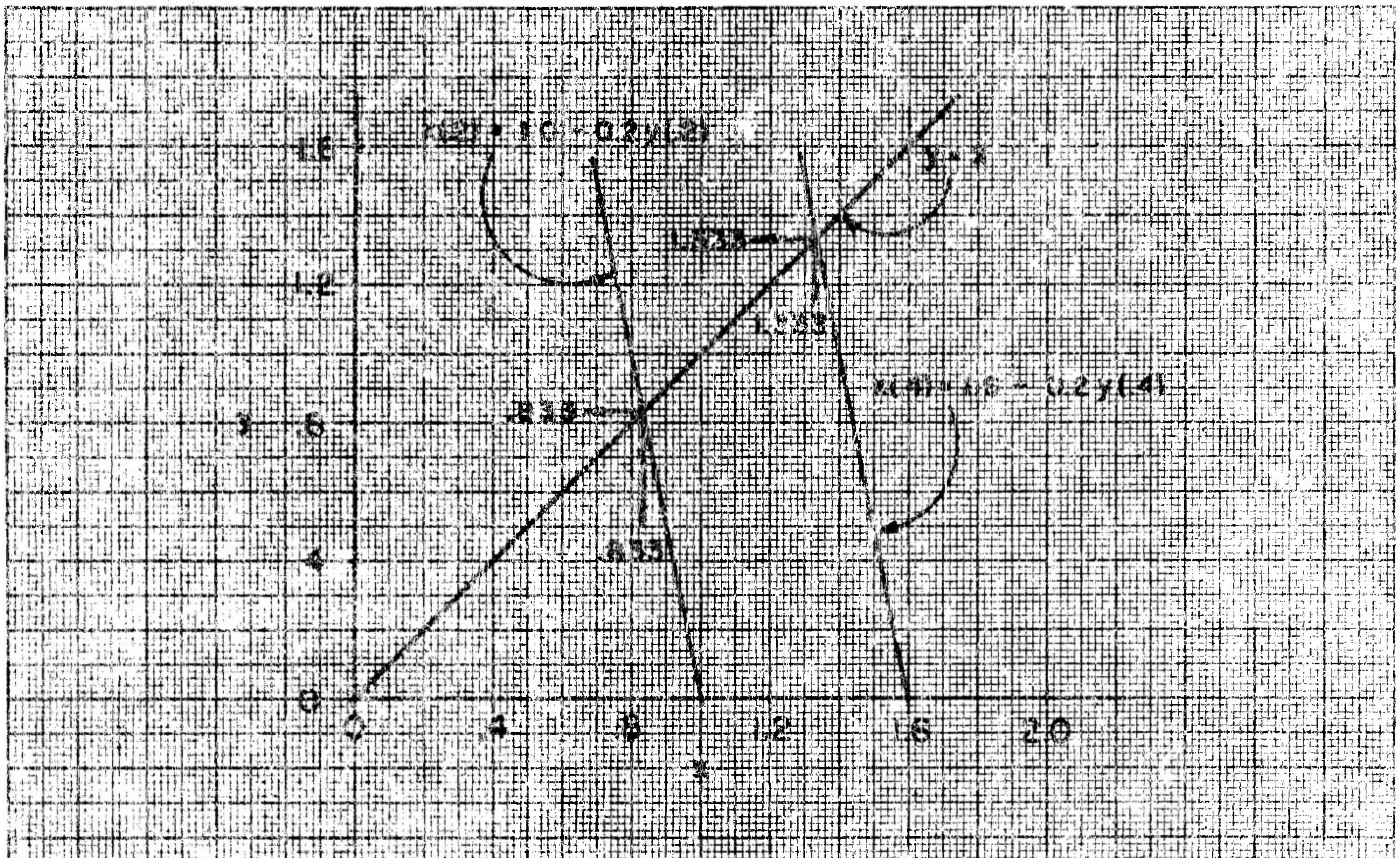


Figure 15. Graphical Solution for R-L Circuit with Linear Inductor

At the next step, we have

$$s(0.6) = \beta_1 y_2 + \beta_2 y_1 + \beta_3 y_0$$

$$= (0.48)(1.333) + (0.56)(0.833) + (0.64)(0)$$

$$= 1.107 \qquad (54)$$

$$\bar{s}(0.6) = x_1(0.6) - s(0.6)$$

$$= 3.000 - 1.107$$

$$= 1.893 \qquad (55)$$

$$x(0.6) = y(0.6) = 1.578 \qquad (56)$$

The rest of the calculations are made in similar fashion, except that equations like (47), (51), and (54) are handled by the method of Fig. 11 rather than being written out explicitly.

Similar calculations have been made for h = 0.1 and h = 0.5. The results for the three different values of h are given in Table II on the following page, along with the correct values. The ϕ-i relation assumed is such that the inductor has an inductance of one henry for the assumed single turn. The relation between the current and applied voltage is easily shown to be

$$I(s) = \frac{s}{s^2 + 2s + 2} E(s) \qquad (57)$$

For a 5-volt step, it follows that

$$i(t) = 5 e^{-t} \sin t \qquad (58)$$

t	Current			
	Approximate Values			Correct Value
	h = 0.1	h = 0.2	h = 0.5	
0	0	0	0	0
0.1	0.454			0.452
0.2	0.818	0.834		0.814
0.3	1.101			1.095
0.4	1.313	1.333		1.305
0.5	1.461		1.667	1.434
0.6	1.557	1.578		1.550
0.7	1.606			1.600
0.8	1.618	1.635		1.612
0.9	1.599			1.593
1.0	1.552	1.565	1.667	1.548
1.2		1.414		1.404
1.4		1.218		1.215
1.5			1.111	1.113
1.6		1.007		1.009
1.8		0.799		0.805
2.0		0.618	0.556	0.615
2.5			0.186	0.246
3.0			0.000	0.036
3.5			-0.061	-0.053
4.0			-0.062	-0.069
4.5			-0.042	-0.055
5.0			-0.022	-0.032

Table II. Results of Calculations for Series Resonant Circuit with Linear Inductor

A plot of the values in Table II shows that the curve for $h = 0.1$ is practically coincident with the correct curve. The values for $h = 0.2$ differ by not more than 0.030 from the correct values in the range covered, an error of not more than about [illegible] per cent. An increment of $h = 0.5$ leads to fairly big errors but gives a reasonably good qualitative picture: the maximum and minimum points of the curves occur at the proper places. In general, the error varies roughly as h^2.

The same process is employed if the inductor is nonlinear, except that y and x are no longer related by a simple proportion. For the example illustrated here, the curve of Fig. 15 is used. Remembering that $y = i$ and $x = \phi$, it will be seen that this curve is a typical magnetization curve, applicable to a variety of common magnetic materials.

Suppose that $h = 0.2$ is taken for the time increment. The same table is used for the β-sequence and a table like Table II is started, beginning with

$$\overline{3}(0.2) = 1.000 \tag{59}$$

as before. In this case, as indicated by Fig. 15, the values of x and y are

$$\begin{aligned} x(0.2) &= 0.89 \\ y(0.2) &= 0.57 \end{aligned} \tag{60}$$

[If values of current (y) are all that is required, there is no need to record x, since it is not used in the computations.]

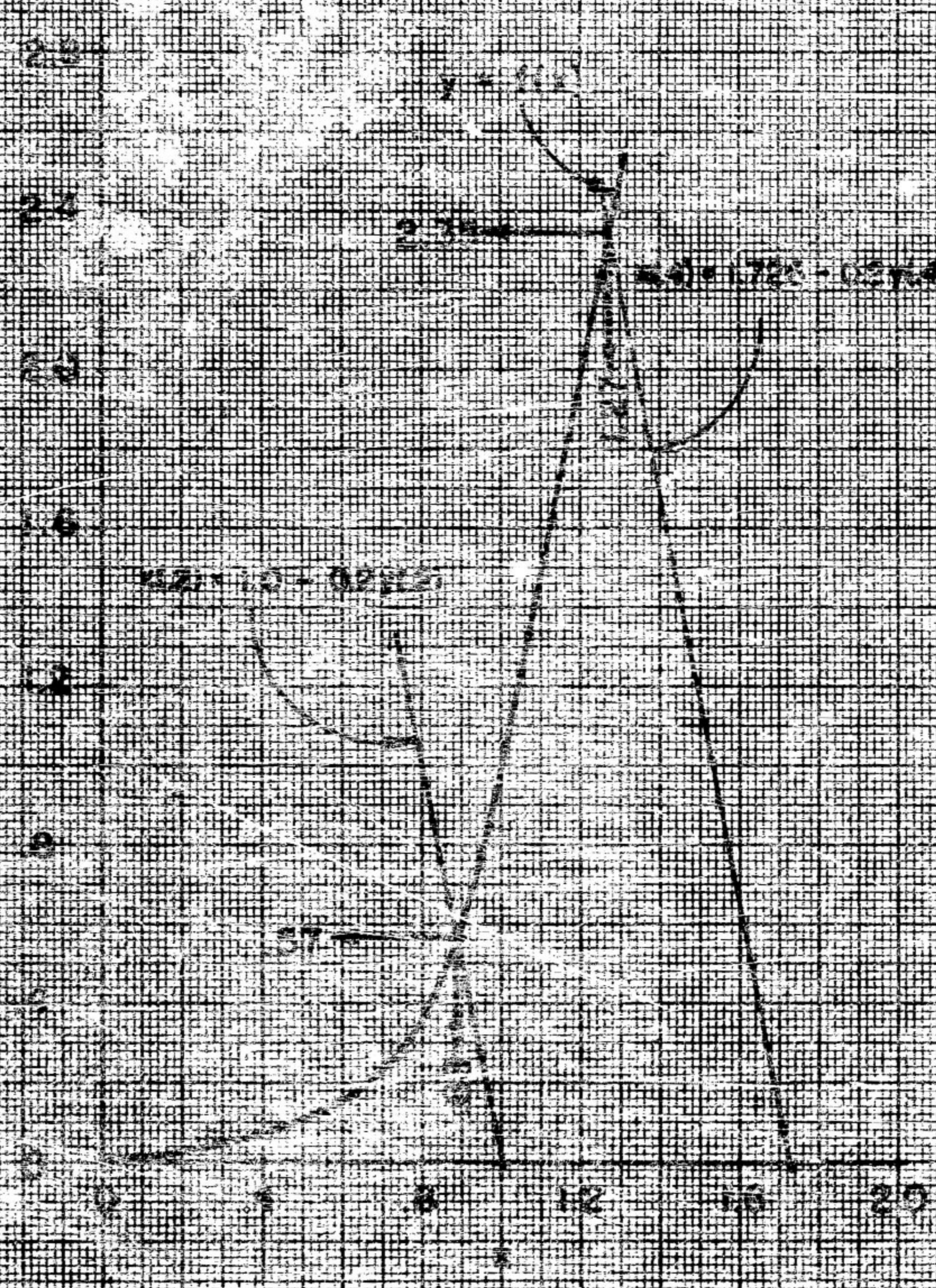


Figure 15. [illegible]

At the next step, the values are

$$S(0.4) = (0.48)(0.57) + (0.56)(0) = 0.274 \tag{61}$$

$$\bar{S}(0.4) = 2.000 - 0.274 = 1.726 \tag{62}$$

$$\left.\begin{aligned} x(0.4) &= 1.26 \\ y(0.4) &= 2.35 \end{aligned}\right\} \quad \text{(from Fig. 15)} \tag{63}$$

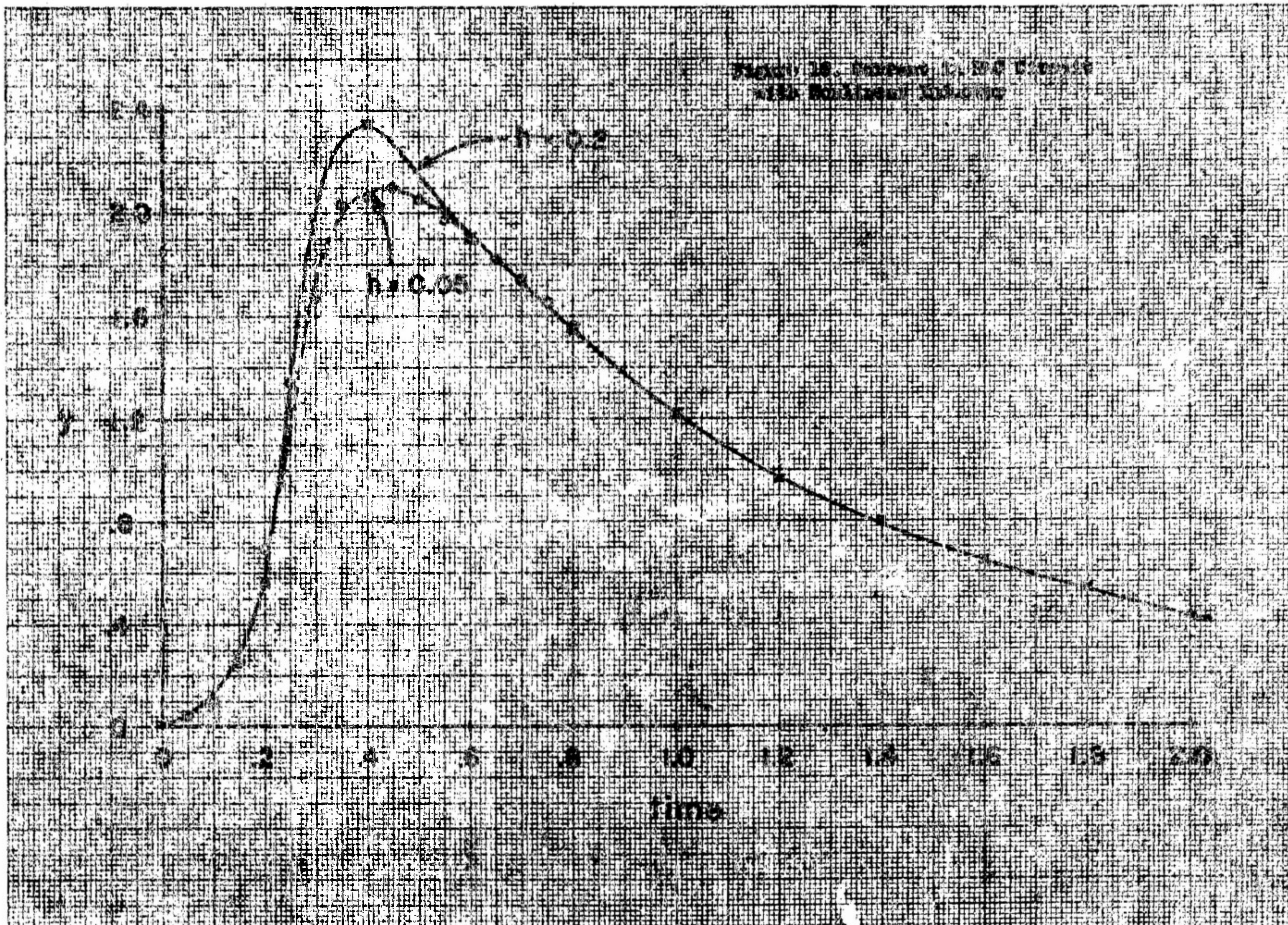
The calculations are continued in similar fashion as far as necessary. Results of such calculations for h = 0.05, 0.1 and 0.2 are given in Table III on the following page and plotted in Fig. 16.

Inspection of the table and the curves indicates that h = 0.2 is not an unreasonable increment, except for one poor value at t = 0.4. For small values of t, the small increments give better results. In all cases there is some fluctuation in the values, caused in part by insbility to obtain more than three significant figures from Fig. 15. Some of the accuracy attainable by the use of the small increments is not realized for this reason.

t	Approximate Current Values		
	h = 0.05	h = 0.1	h = 0.2
0	0	0	0
0.1	0.10	0.10	
0.2	0.69	0.64	0.57
0.3	1.67	1.80	
0.4	2.06	2.11	2.35
0.45	2.10		
0.5	2.05	2.06	
0.6	1.90	1.88	1.90
0.7	1.74	1.74	
0.8	1.55	1.56	1.55
0.9		1.38	
1.0		1.25	1.22
1.1		1.12	
1.2		0.98	0.97
1.3		0.90	
1.4		0.80	0.80
1.5		0.73	
1.6		0.65	0.65
1.7		0.60	
1.8		0.54	0.54
1.9		0.48	
2.0		0.44	0.43

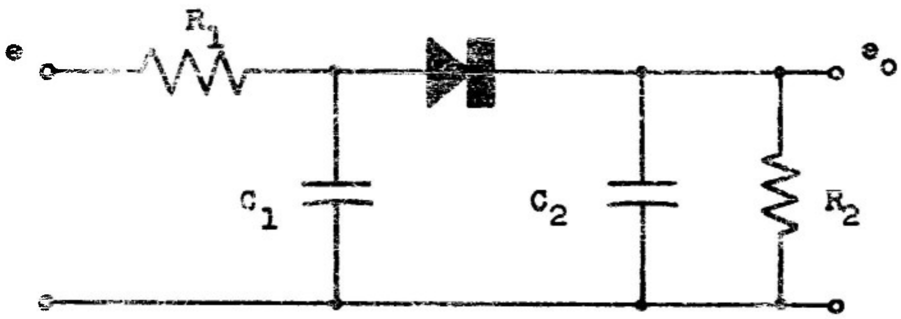
Table III. Results of Calculations for Series Resonant Circuit with Nonlinear Inductor

Fig. 18
p. 35

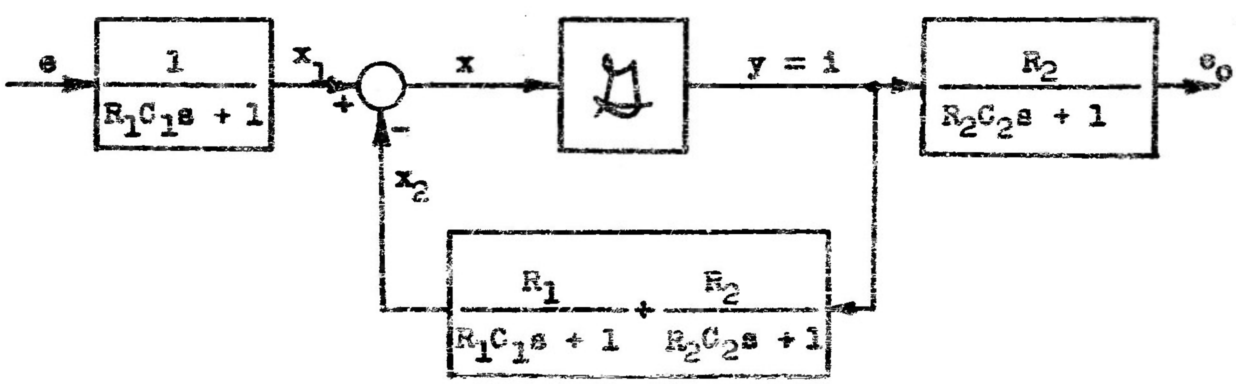


(2) RC Ladder Network Containing a Diode

The circuit diagram of the RC ladder network considered in the second example is shown in Fig. 17(a); the block diagram[1] is given in Fig. 17(b).



(a)



(b)

Figure 17. RC Ladder Network and Block Diagram

In this case, the diode is described by a conductance function, denoted by $\ddot{\mathfrak{D}}$. The variable y is the current through the diode.

For simplicity, it is assumed that $R_1 = R_2 = 1$ and that $C_1 = C_2 = 1$. A 3-volt step is applied at the input terminals of the network with the capacitors initially uncharged.

With these values, the preliminary calculations are as follows. The transform of the input voltage is

$$e(s) = \frac{3}{s} \tag{64}$$

and the transfer function of the input block is

$$G_1(s) = \frac{1}{s+1}, \tag{65}$$

with the result that

$$x_1(s) = \frac{3}{s\,(s+1)} \tag{66}$$

$$x_1(t) = 3\,(1 - e^{-t}) \tag{67}$$

The transfer function of the feedback block is

$$\beta(s) = \frac{2}{s+1}, \tag{68}$$

so that

$$\beta(t) = 2\,e^{-t}. \tag{69}$$

The β-sequence is therefore

$$\frac{0.4000}{2},\ 0.3274,\ 0.2682,\ 0.2196,\ 0.1798,\ \cdots \tag{70}$$

if $h = 0.20$.

It may also be noted that, for the particular values selected,

$$e_o(s) = \frac{x_2(s)}{2}, \tag{71}$$

and therefore

$$e_o(t) = \frac{x_2(t)}{2}. \tag{72}$$

A linear case is considered first, as a check on the accuracy of the method. If a 1-ohm resistor is substituted for the diode,

$$x = y \tag{73}$$

Proceeding as before, Table IV is obtained.

t	x_1	S	$\bar{S}$	$x=y$	x_2	e_o
0	0	-	-	0	0	0
0.2	.5438	.0000	.5438	.4532	.0906	.0453
0.4	.9890	.1484	.8406	.7005	.2885	.1442
0.6	1.3536	.3509	1.0027	.8356	.5180	.2590
0.8	1.6520	.5610	1.0910	.9092	.7428	.3714
1.0	1.8964	.7571	1.1393	.9494	.9470	.4735
1.2	2.0964	.9308	1.1656	.9713	1.1251	.5625
1.4	2.2602	1.0802	1.1800	.9833	1.2769	.6384
1.6	2.3943	1.2064	1.1879	.9899	1.4044	.7022
1.8	2.5041	1.3118	1.1923	.9936	1.5105	.7553
2.0	2.5940	1.3993	1.1947	.9956	1.5984	.7992

Table IV. Calculations for RC Ladder Network with Linear Resistor

Similar calculations were made for $h = 0.1$ and $h = 0.5$. The results of three series of calculations are summarized in Table V, which also includes the correct values computed from the relation

$$e_o(t) = 1 - \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3t} \tag{74}$$

t	Output Voltage			
	Approximate Values			Correct Value
	h = 0.1	h = 0.2	h = 0.5	
0	0	0	0	0
0.1	0.0130			0.0132
0.2	0.0461	0.0453		0.0463
0.3	0.0918			0.0921
0.4	0.1449	0.1442		0.1451
0.5	0.2016		0.1967	0.2018
0.6	0.2594	0.2590		0.2594
0.7	0.3163			0.3164
0.8	0.3713	0.3714		0.3714
0.9	0.4238			0.4232
1.0	0.4732	0.4735	0.4752	0.4731
1.2		0.5625		0.5619
1.4		0.6384		0.6376
1.5			0.6762	0.6709
1.6		0.7022		0.7013
1.8		0.7553		0.7543
2.0		0.7992	0.8046	0.7982
2.5			0.8847	0.8771
3.0			0.9329	0.9253
3.5			0.9617	0.9547

Table V. Results of Calculations for RC Ladder Network with Linear Resistor

An an approximation to the voltage-current characteristic of a crystal diode, the relation

$$
\begin{aligned}
i &= 0 \qquad & e < 0 \\
&= e^4 \qquad & e > 0
\end{aligned}
\tag{75}
$$

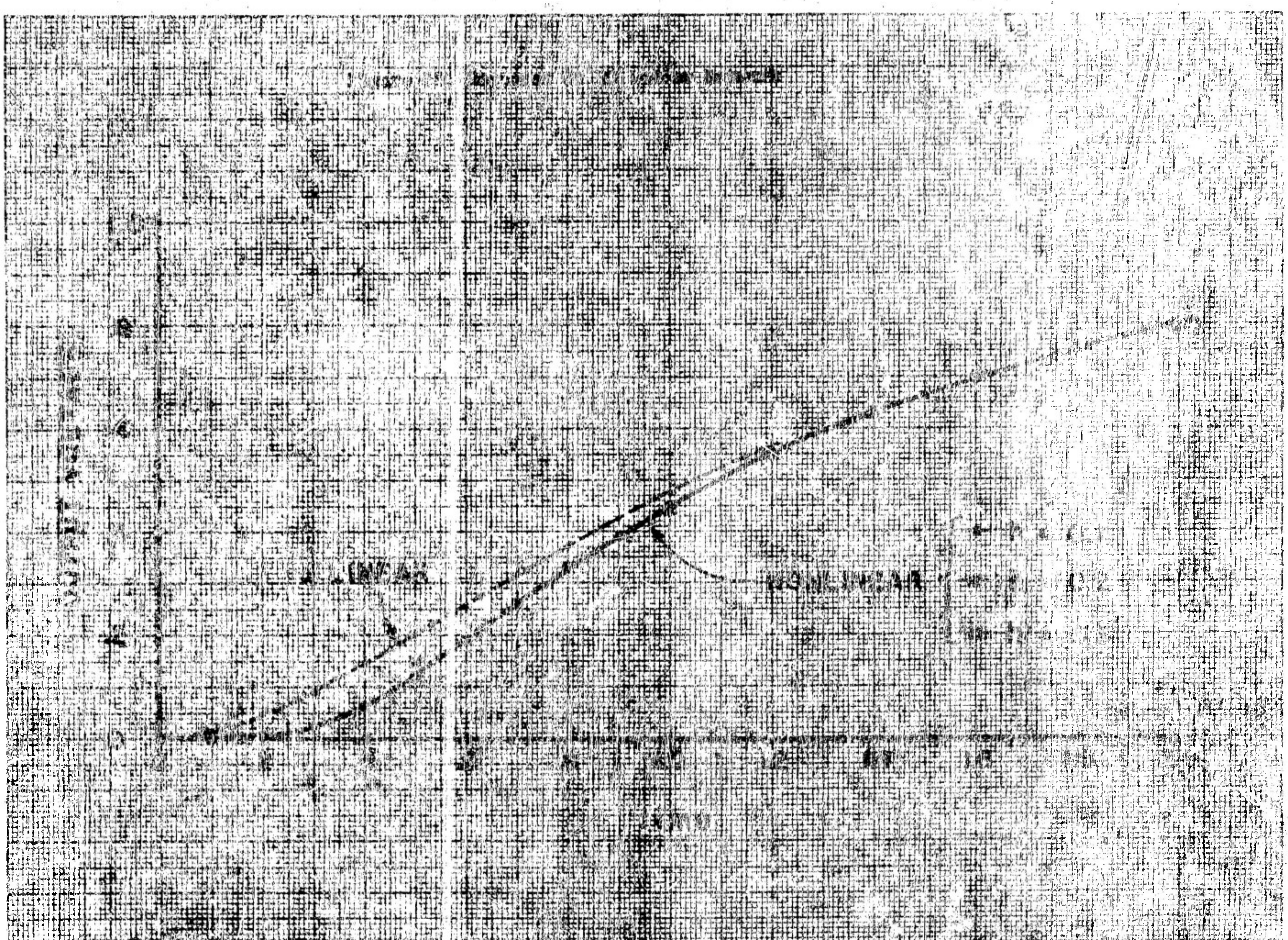
was used. In the general notation used previously,

$$
y = x^4 \qquad x > 0 \tag{76}
$$

Equation (76) was plotted and the calculations were repeated, using the graphical method described before. Results for h = 0.1, 0.2, and 0.5 are given in Table VI and plotted in Fig. 18.

t	Approximate Output Voltage		
	h = 0.1	h = 0.2	h = 0.5
0	0	0	0
0.1	0.000_3		
0.2	0.005	0.008	
0.3	0.022		
0.4	0.062	0.068	
0.5	0.121		0.150
0.6	0.190	0.192	
0.7	0.261		
0.8	0.329	0.328	
0.9	0.392		
1.0	0.450	0.450	0.442
2.0		0.800	0.800

Table VI. Results of Calculations for RC Ladder Network with Nonlinear Diode



In the linear case, the solution for h = 0.1 differs from the correct solution by not more than three units in the fourth decimal place. For h = 0.2 and 0.5, the maximum errors are 10 and 80 units, respectively. As before, the error varies roughly as h^2.

Since a correct solution to the nonlinear problem is not available, the error cannot be determined precisely. However, it appears that the results for h = 0.1 are about as good as can be expected with the number of significant figures employed.

(3) Torque-Saturated Servomechanism

As another example, the simple torque-saturated servomechanism shown in Fig. 19 is considered.

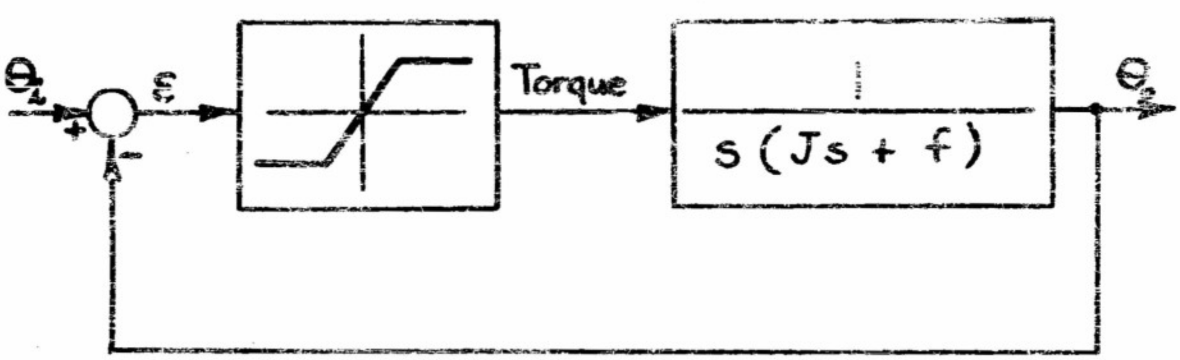


Figure 19. Block Diagram of Torque-Saturated Servomechanism

The block diagram of the torque-saturated servomechanism is reduced to the general form by moving the linear block to the right of the take-off point and into the feedback path; the transfer functions are then

$$G_1(s) = 1 \qquad (77)$$

$$\beta(s) = G_2(s) = \frac{1}{s\,(Js + f)} \qquad (78)$$

The relation between error and torque may be expressed as

$$T = \begin{cases} K\,\mathcal{E}_c & \mathcal{E} > \mathcal{E}_c \\ K\,\mathcal{E} & -\mathcal{E}_c < \mathcal{E} < \mathcal{E}_c \\ -K\,\mathcal{E}_c & \mathcal{E} < -\mathcal{E}_c \end{cases} \qquad (79)$$

Without loss of generality, a new set of dimensionless parameters may be introduced; a suitable choice is[22]

$$\zeta = \frac{f}{2\sqrt{J\,K}} \qquad (80)$$

$$\omega_o = \sqrt{\frac{K}{J}} \qquad (81)$$

$$\tau = \omega_o\,t \qquad (82)$$

$$x = \frac{\mathcal{E}}{\mathcal{E}_c} \qquad (83)$$

In the example which follows, we have chosen to let $\zeta = 0.5$, $\omega_o = 1$, and $\mathcal{E}_c = 1$. This is equivalent to selecting $K = 1$, $J = 1$, and $f = 1$ in the original system, and leads to

$$\tau = t \qquad (84)$$

$$x = \mathcal{E} \qquad (85)$$

It will be assumed that a 5-unit step is applied to the input with the system initially at rest. For the values used, we have

$$x_1(t) = 5 \tag{86}$$

$$\beta(t) = 1 - e^{-t} \tag{87}$$

$$\begin{aligned} x_2(t) &= e_o(t) \\ &= \theta_c(t) \end{aligned} \tag{88}$$

For $h = 0.4$, the β-sequence becomes

$$0.0000,\ 0.1319,\ 0.2203,\ 0.2795,\ 0.3192,\ \cdots \tag{89}$$

The necessity for the graphical solution of two algebraic equations is eliminated, since $\beta(0) = 0$. In this particular situation, the value of y at any time is dependent only on the past values of x and can be calculated without a knowledge of the present value of y. The calculations are started as follows:

At $t = 0$:

$$\begin{aligned} x_1(0) &= 5.0000 \\ x_2(0) &= 0.0000 \\ x(0) &= 5.0000 \\ y(0) &= 1.0000 \end{aligned} \tag{90}$$

At $t = 0.4$:

$$\begin{aligned} x_2(.4) &= (0.1319)\left(\frac{1.0000}{2}\right) \\ &= 0.0660 \\ x(.4) &= 4.9340 \\ y(.4) &= 1.0000 \end{aligned} \tag{91}$$

At $t = 0.8$:
$$x_2(.8) = (0.1319)(1.0000) + (0.2203)\left(\frac{1.0000}{2}\right) = 0.2421$$
$$x(.8) = 4.7579 \qquad (92)$$
$$y(.8) = 1.0000$$

These calculations are best carried out in tabular form; part of the work is given in Table VII. As a reminder that $y(0)/2$ is used in the calculation of $x_2(t)$, the correct value of y at $t = 0$ has been crossed out and one-half of the correct value entered at this point in the table.

t	x_1	x_2	$x = \mathcal{E}$	$y = T$
0	5.0000	0	5.0000	0.5000 ~~1.0000~~
0.4	"	0.0660	4.9340	1.0000
0.8	"	0.2421	4.7579	1.0000
1.2	"	0.4920	4.5080	1.0000
1.6	"	0.7908	4.2092	1.0000
2.0	"	1.1228	3.8772	1.0000
2.4	"	1.4776	3.5224	1.0000
2.8	"	1.8471	3.1529	1.0000
3.2	"	2.2269	2.7731	1.0000
3.6	"	2.6132	2.3868	1.0000
4.0	"	3.0042	1.9958	1.0000
4.4	"	3.3981	1.6019	1.0000
4.8	"	3.7940	1.2060	1.0000
5.2	"	4.1912	0.8088	0.8088

Table VII. Calculations for Torque-Saturated Servomechanism

In Fig. 20, the results of the calculations have been superimposed on a curve which was determined analytically[22]. For $t < 6$, the approximate values obtained by the step-by-step method are consistently higher than the correct values; the difference is not more than 140 units in the fourth decimal place. In a separate set of calculations, not included here, $h = 0.2$ was used; the difference between the approximate and correct values does not exceed 40 units. For $t > 6$, the accuracy is somewhat better.

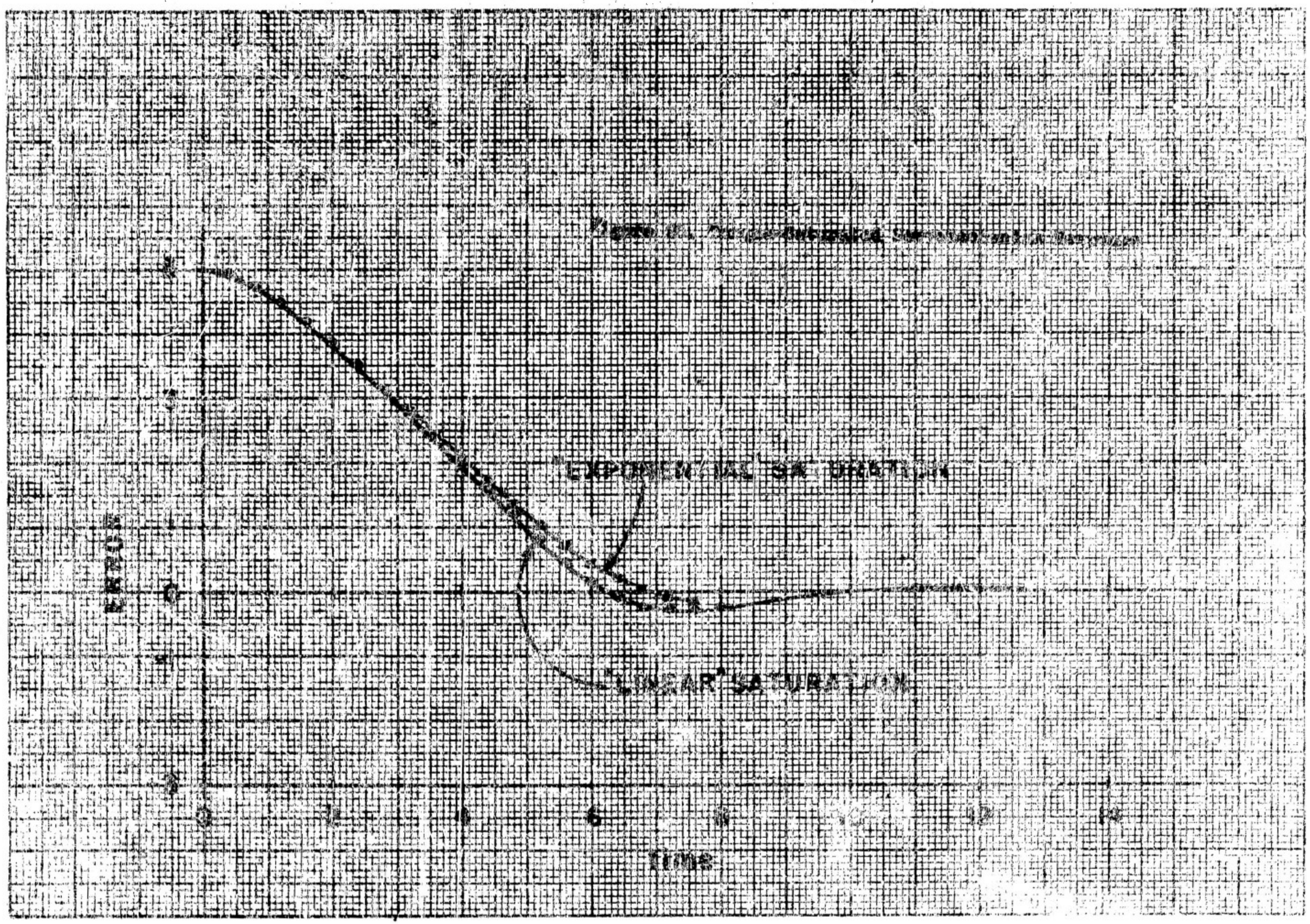
A series of calculations was also made for an error-torque relation which may be expressed as

$$T = K\,\mathcal{E}_c\left[1 - e^{-\left(\frac{\mathcal{E}}{\mathcal{E}_c}\right)}\right]_{\mathcal{E} > 0} \qquad (93)$$

and

$$T(-\mathcal{E}) = -T(\mathcal{E}) \qquad (94)$$

This expression for the torque reduces to $K\,\mathcal{E}$ for small errors and approaches $K\,\mathcal{E}_c$ for large errors, like the idealized "linear" or "straight-line" saturation characteristic used in the earlier calculations. The resulting curve for the "exponential" saturation characteristic is also plotted in Fig. 20. There is very little difference in the curves for large errors; in the neighborhood of zero error, the linear saturation characteristic leads to a larger overshoot, as might be expected.


EXPONENTIAL SATURATION
"LINEAR" SATURATION
ERROR
time
0
2
4
6
8
12
14

(4) RC Network Containing a Diode

The final example in this report is the RC network shown in Fig. 21(a). If the block diagram is contructed and reduced to the general form, Fig. 21(b) is obtained. Since $\beta(s)$ has a numerator of higher order than the denominator, it will be advantageous to interchange the forward and feedback blocks by use of the theorem described on page 12; the resulting block diagram is given in Fig. 21(c).

The step-by-step calculation process for systems having a block diagram of the type given in Fig. 21(c) is similar to the process already described, but differs in some details. A general discussion of the required modifications in the method will therefore be presented before this specific case is treated.

For a feedback system with the nonlinear block in the feedback path, the basic equations are

$$x_2(nh) = x_1(nh) - x(nh) \tag{95}$$

$$x(nh) = g\left[\, y(nh)\right] \tag{96}$$

$$y(nh) = \int_0^{t=nh} x_2(t-\tau)\, w(\tau)\, d\tau \tag{97}$$

$$\cong \frac{h\, w_0}{2}\, x_2(nh) + Q(nh) \tag{98}$$

where $$w(t) = \mathcal{L}^{-1}\left[\frac{1}{\beta(s)}\right], \tag{99}$$

and Q(nh), like the S(nh) used previously, denotes a weighted sum of past values of x_2, up to $x_2\left[(n-1)h\right]$, obtained as part of the approximation to the convolution integral.

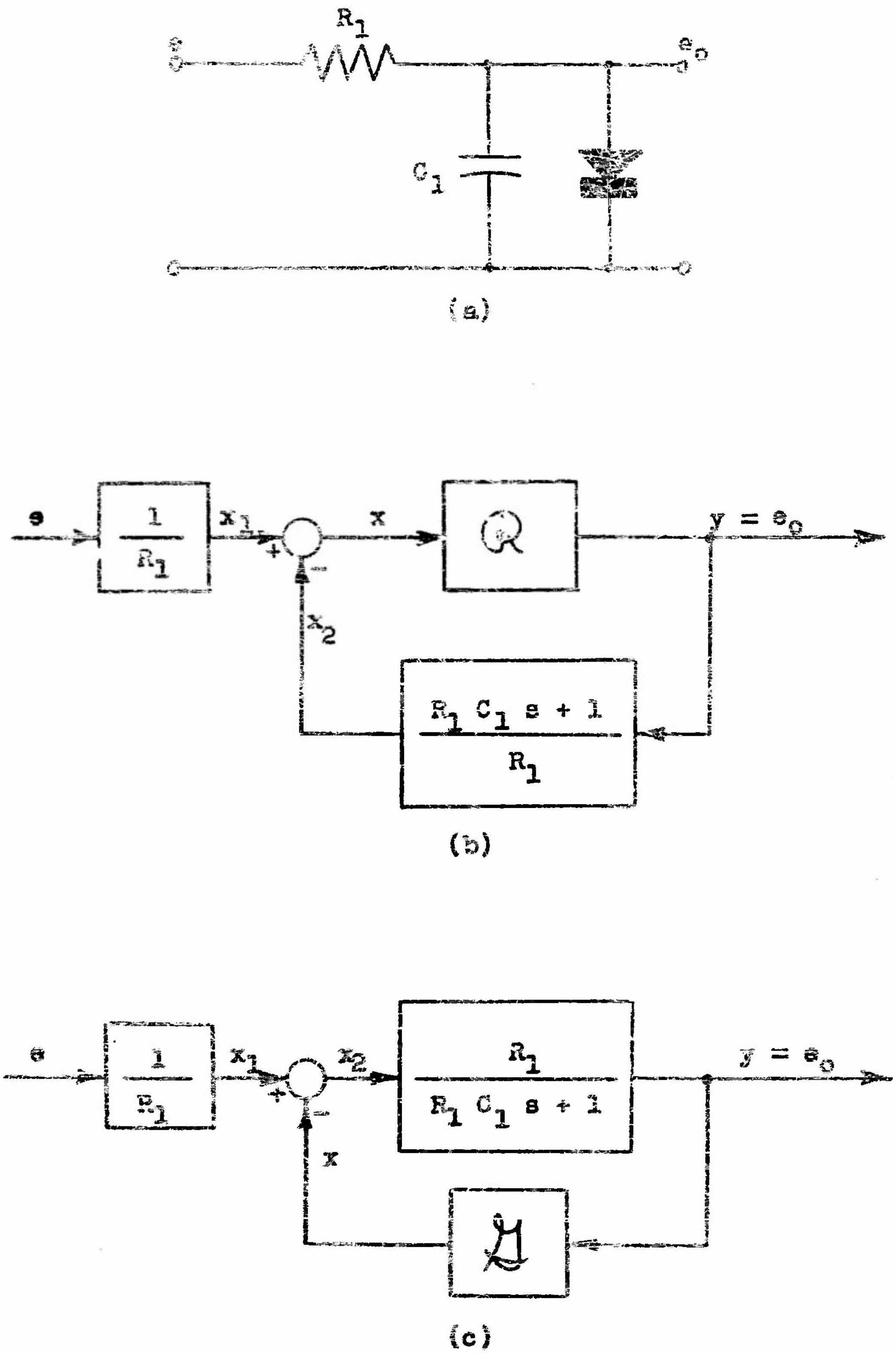


Figure 21. RC Network and Block Diagrams

Combining Eqs. (95) and (98) and setting $hW_o/2 = \rho$, we obtain

$$y(nh) = \rho\left[x_1(nh) - x(nh)\right] + Q(nh) \qquad (100)$$

$$= \bar{Q}(nh) - \rho\, x(nh) \qquad (101)$$

where $$\bar{Q}(nh) = Q(nh) + \rho\, x_1(nh) \qquad (102)$$

Equations (96) and (101) constitute a pair of simultaneous algebraic equations in x and y which can be solved by a graphical process illustrated in Fig. 22(a). Alternatively, to permit use of a plot of y as a function of x, the axes can be interchanged as indicated in Fig. 22(b).

For this example, it will be assumed that $R_1 = 1$, $C_1 = 1$, and the input is a 2-volt step, applied with the capacitor initially uncharged. A linear case for which x = y, obtained by replacing the diode by a linear 1-ohm resistor, will be treated first. With these particular values and using h = 0.2, we have:

$$x_1(t) = 2.0000 \qquad (103)$$

$$W(s) = \frac{1}{\beta(s)} = \frac{1}{s+1} \qquad (104)$$

$$W(t) = e^{-t} \qquad (105)$$

$$\rho = \frac{h\,W_o}{2} = 0.1000 \qquad (106)$$

$$\rho x_1 = 0.2000 \qquad (107)$$

The W-sequence, including the factor h, is

$$\frac{0.2000}{2},\ 0.1637,\ 0.1341,\ 0.1098,\ 0.0899,\ \cdots \qquad (108)$$

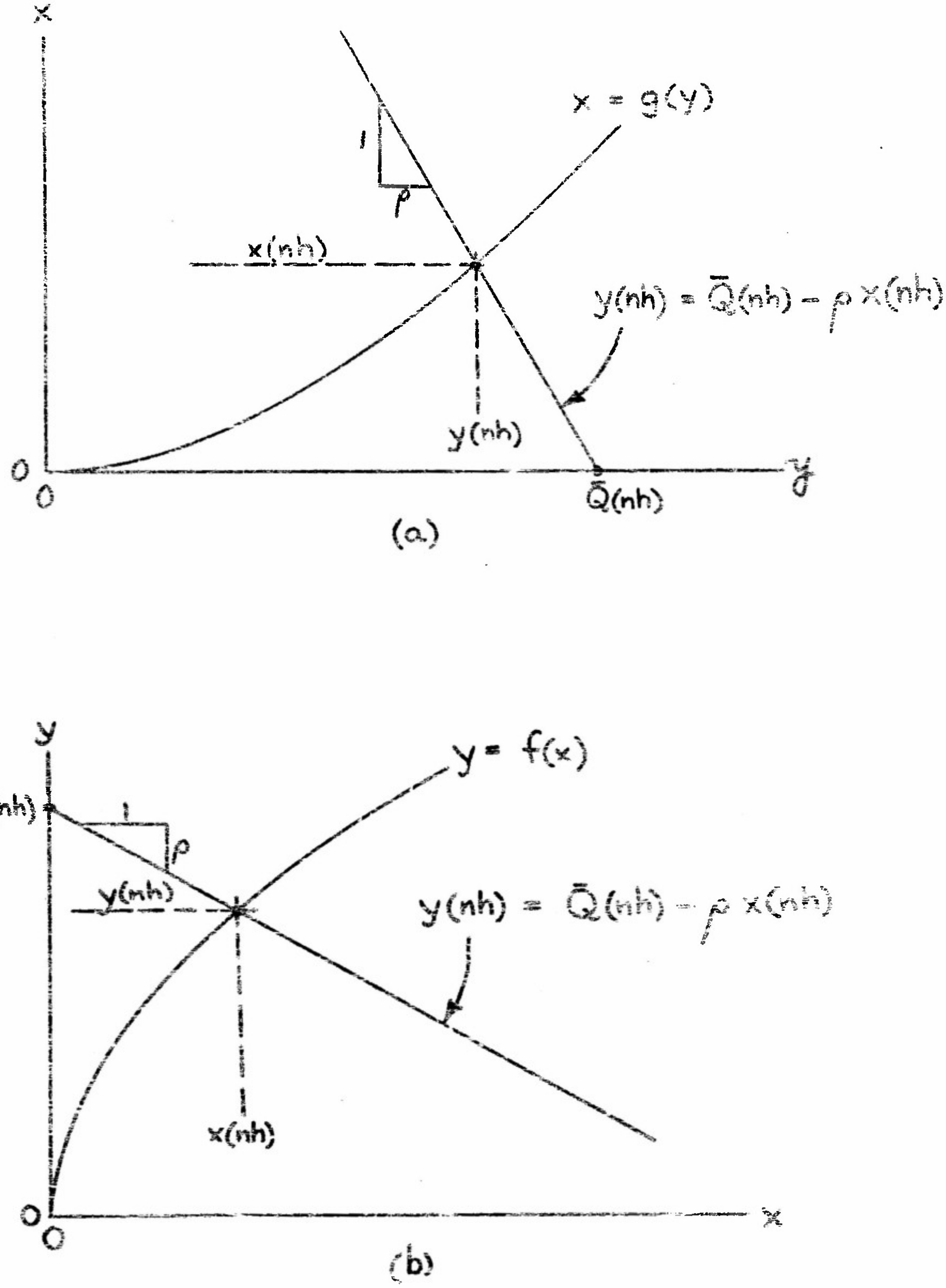


Figure 22. Alternative Graphical Solutions for x(nh) and y(nh)

The calculations proceed as follows. At $t = 0$:

$$y(0) = x(0) = 0$$

$$x_2(0) = x_1(0) - x(0) \tag{109}$$

$$= 2.0000$$

At $t = 0.20$: $$Q(0.2) = \beta_1 \left(\frac{x_2(0)}{2}\right)$$

$$= (0.1637)(1.0000)$$

$$= 0.1637$$

$$\tilde{Q}(0.2) = Q(0.2) + \rho\, x_1(0.2)$$

$$= 0.1637 + 0.2000$$

$$= 0.3637 \tag{110}$$

Using the graph, $x(0.2) = y(0.2) = 0.3306$

$$x_2(0.2) = x_1(0.2) - x(0.2)$$

$$= 2.0000 - 0.3306$$

$$= 1.6694$$

At $t = 0.40$: $$Q(0.4) = \beta_2 \left(\frac{x_2(0)}{2}\right) + \beta_1\, x_2(0.2)$$

$$= (0.1341)(1.0000) + (0.1637)(1.6694)$$

$$= 0.4074$$

$$\tilde{Q}(0.4) = 0.4074 + 0.2000$$

$$= 0.6074$$

Using the graph, $x(0.4) = y(0.4) = 0.5522$ (111)

$$x_2(0.4) = 2.0000 - 0.5522$$

$$= 1.4478$$

The rest of the calculations are carried out in similar fashion; the work is summarized in Table VIII. In this table, as in Table VII, the value of $y(0)$ is crossed out and replaced by $y(0)/2$ to facilitate the subsequent multiplications which determine the values of Q.

t	x_1	ρx_1	Q	$\bar{Q}$	$x = y$	x_2
0	2.0000	0.2000	------	------	0.0000	1.0000 ~~2.0000~~
0.2	2.0000	0.2000	0.1637	0.3637	0.3306	1.6694
0.4	2.0000	0.2000	0.4074	0.6074	0.5522	1.4478
0.6	2.0000	0.2000	0.5707	0.7707	0.7006	1.2994
0.8	2.0000	0.2000	0.6801	0.8801	0.8001	1.1999
1.0	2.0000	0.2000	0.7533	0.9533	0.8666	1.1334
1.2	2.0000	0.2000	0.8023	1.0023	0.9112	1.0888
1.4	2.0000	0.2000	0.8351	1.0351	0.9410	1.0590
1.6	2.0000	0.2000	0.8571	1.0571	0.9610	1.0390
1.8	2.0000	0.2000	0.8717	1.0717	0.9743	1.0257
2.0	2.0000	0.2000	0.8815	1.0815	0.9832	1.0168

Table VIII. Calculations for RC Network with Linear Resistor

The results of similar calculations for h = 0.1 and h = 0.4 are given in Table IX, along with the correct values computed from the equation

$$y(t) = 1 - e^{-2t} \tag{112}$$

t	Output Voltage			
	Approximate Values			Correct Value
	h = 0.1	h = 0.2	h = 0.4	
0	0	0	0	0
0.1	0.1814			0.1813
0.2	0.3300	0.3306		0.3297
0.3	0.4516			0.4512
0.4	0.5511	0.5522	0.5568	0.5507
0.5	0.6326			0.6321
0.6	0.6993	0.7006		0.6988
0.7	0.7540			0.7534
0.8	0.7987	0.8001	0.8055	0.7981
0.9	0.8353			0.8347
1.0	0.8653	0.8666		0.8647
1.2		0.9112	0.9168	0.9093
1.4		0.9410		0.9392
1.6		0.9610	0.9665	0.9592
1.8		0.9743		0.9727
2.0		0.9832	0.9887	0.9817

Table IX. Results of Calculations for RC Network with Linear Resistor

The errors at $t = 0.8$ are 6, 20, and 74 units in the fourth decimal place; the variation is again roughly as h^2. In this case, the values obtained using $h = 0.4$ appear to be quite satisfactory for plotting purposes.

A similar set of calculations was made for this network with a nonlinear diode, described by the relation

$$i = e^4 \tag{113}$$

Since y is identified with output voltage and x with current, Eq. (113) becomes

$$x = y^4 \tag{114}$$

in the general notation. Following the scheme of Fig. 22(b), the graphical solution looks exactly the same as in Fig. 15, except that x and y are interchanged. The results are given in Table X.

In this problem, the calculations with $h = 0.4$ appear to exhaust the accuracy of the graphical solution. Use of smaller steps does not increase the accuracy, since the main source of error is the graphical determination of x and y. The smaller steps do, of course, give a more complete picture of the response.

The results are also plotted in Fig. 23, using steps of $h = 0.1$ for $t < 1.0$ and $h = 0.2$ for $t > 1.0$. The correct (computed) curve for the linear case is included for comparison.

t	Approximate Output Voltage		
	h = 0.1	h = 0.2	h = 0.4
0	0	0	0
0.1	0.190		
0.2	0.362	0.362	
0.3	0.513		
0.4	0.643	0.643	0.636
0.5	0.757	[illegible]	
0.6	0.825	0.830	
0.7	0.890		
0.8	0.930	0.925	0.930
0.9	0.950		
1.0	0.970		
1.2		0.993	0.995
1.4		0.998	
1.6		1.000	1.000
1.8		1.000	
2.0		1.000	1.010

Table X. Results of Calculations for RC Network with Nonlinear Diode

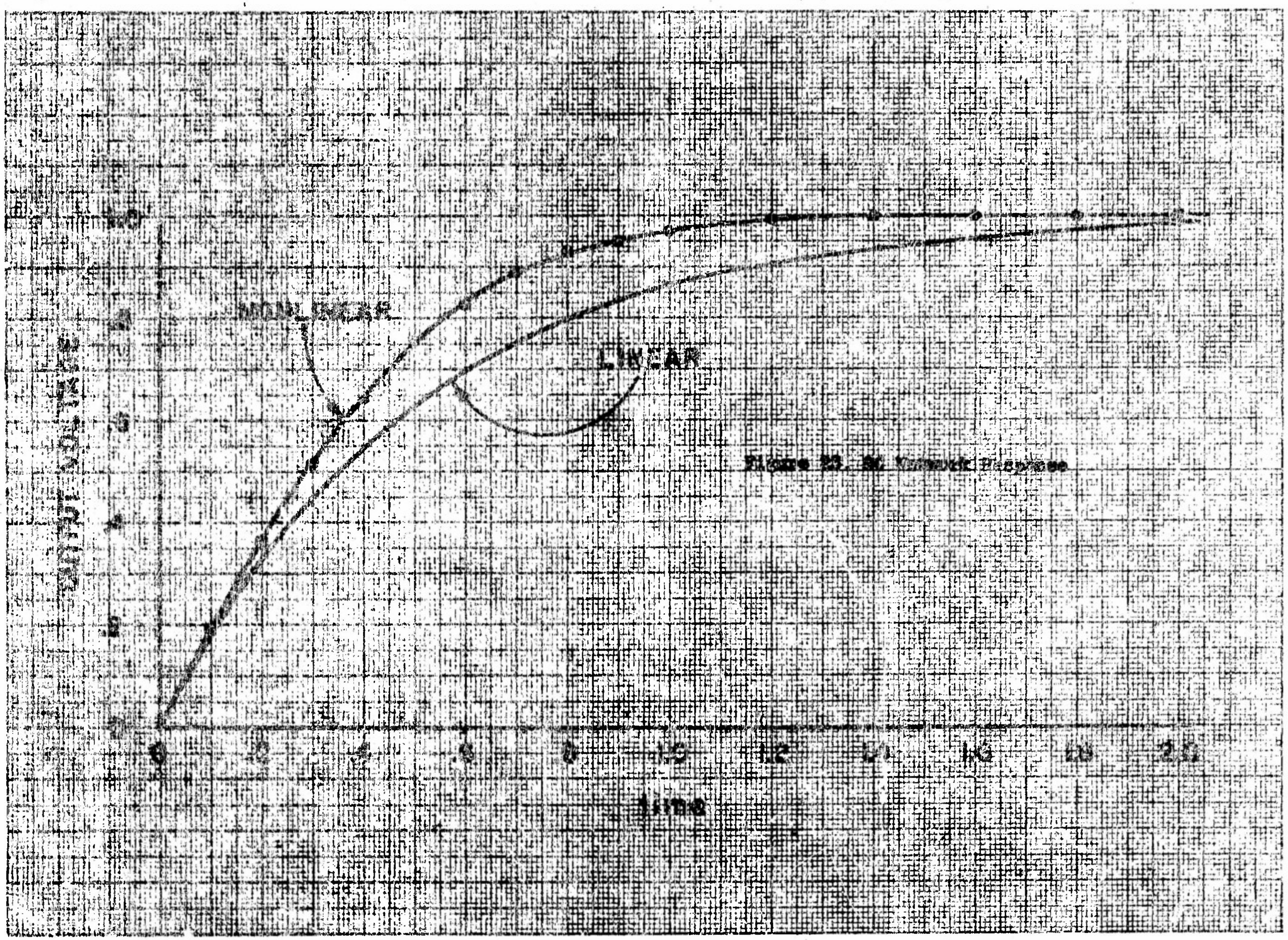


Figure 23. [illegible] Transient Response

Errors

The accuracy of the results obtained by the procedures outlined in this report is affected by a variety of possible errors:

(1) Calculation errors - due to arithmetic mistakes in carrying out the various multiplications, additions, and subtractions

(2) Graphical errors - due to inability to specify the intersection of two curves more closely than three or four significant figures

(3) Round-off error

(4) Integration error - due to inadequacies of the approximate integration formula (trapezoidal rule), assuming that the values of the integrand are correct

(5) Inherited error - due to inaccuracies in the integrand resulting from previous errors

The possibility of errors of the first three types can be minimized by elementary strategems: (1) careful work and difference checks of the variables as they are obtained; (2) use of a sharp pencil and a large piece of graph paper; and (3) carrying more decimal places than will be retained in the final results. The other types of error deserve a more detailed discussion.

Because the trapezoidal rule is based on an approximation of the integrand by a series of straight line segments, it will give correct results for any integrand having a constant first derivative; any errors which are encountered can thus be traced to the higher derivatives of the integrand. A thorough analysis of the question leads to the following estimate of the error:[13]

$$E \cong - \frac{L h^2}{12} z''_m \qquad (115)$$

Where h = size of increment between values;

L = nh = total interval of integration;

z = function to be integrated (integrand);

z''_m = maximum value of the second derivative of the integrand in the interval $0 < t < nh$.

A more optimistic estimate of the error is[12,15]

$$E \cong - \frac{h^2}{12} \left[z'(nh) - z'(0) \right] \qquad (116)$$

where $z'(nh)$ and $z'(0)$ are the first derivatives of the integrand at the end and beginning of the interval. Since the first derivative is the integral of the second, it is true that

$$z'(nh) - z'(0) = \int_0^{nh} z'' \, dt$$

$$= L \, z''_{av} \qquad (117)$$

where z''_{av} denotes the average value of the second derivative of the integrand in the interval under consideration. The two formulas are therefore essentially the same. The minus sign in the formulas reflects the fact that the approximate value of the integral will be too high if the curve which represents the integrand is concave upward, placing the straight-line approximation above the curve.

The integral to be evaluated is either

$$x_2(mh) = \int_0^{mh} \beta(\tau)\, y(t - \tau)\, d\tau \tag{19}$$

or

$$y(mh) = \int_0^{mh} W(\tau)\, x_2(t - \tau)\, d\tau \tag{97}$$

depending on the nature of the block diagram. If, for example, we define

$$z(\tau) = \beta(\tau)\, y(t - \tau), \tag{118}$$

we then have

$$z'(\tau) = -\beta(\tau)\, y'(t - \tau) + \beta'(\tau)\, y(t - \tau) \tag{119}$$

and

$$z''(\tau) = \beta(\tau)\, y''(t - \tau) - 2\beta'(\tau)\, y'(t - \tau) + \beta''(\tau)\, y(t - \tau), \tag{120}$$

where the primes denote differentiation with respect to τ. While $\beta(t)$ or $W(t)$ is known in advance, $y(t)$ and $x_2(t)$ are determined in the course of the calculations and are subject to the errors which are to be estimated. Any estimate of the error must therefore be made after the calculations have been completed. Equation (119), in particular, can be used to obtain a quick estimate of the error as the calculations progress.

Perhaps the most useful conclusion to be drawn from Eqs. (115) and (116) is the variation of error with h. If two series of values have been obtained for a given variable by using two different values of h, and if these values are not subject to other errors, the fact that the error varies as h^2 might be used to pick a third value of h which would lead to results of any desired accuracy.

Although the question of inherited errors has not been thoroughly studied, the following observations can be made. Suppose that the system is described by the relations

$$x(t) = x_1(t) - x_2(t) \tag{8}$$

$$y(t) = f\left[x(t)\right] \tag{9}$$

$$x_2(t) = \int_0^t \beta(\tau)\, y(t-\tau)\, d\tau \tag{19}$$

The value of $x_2(t)$ is approximated by a weighted sum of present and past values of y. If the approximate integration process introduced no error except that due to inaccurate values of y, the error in x_2 would be a weighted sum of the errors in y. Assuming that f(x) is a well-behaved function and that h is small enough so that there are only small changes in $x_1(t)$ and $\beta(t)$ in any increment, there is a tendency for errors in x and y to alternate in sign. This tendency is evident in some of the examples and may be explained as follows. Suppose that one of the values of y is too large; the corresponding value of x_2 from Eq. (19) will likewise be too large. The next value of x, computed from Eq. (8), will be too small and will lead to a value of y, from Eq. (9), which will therefore be small. If the errors are not too big, the process thus tends to be partially self-correcting, with the result that inherited errors have not been a problem.

If the step-by-step process is to be applied to physical systems and checked experimentally, other sources of error are introduced; the most important is the possibility that dynamic behavior of nonlinear elements is not predictable from the static characteristics.

Labor Required

In addition to knowing about the accuracy of a step-by-step method, the user also wants an estimate of the amount of computational labor required to complete a solution. In the method described here, the k^{th} step in the calculation will require about $(3 + k)$ operations, either multiplications, additions, or subtractions, and one determination of values from a curve or equation. Summing over n steps, the number of operations is estimated to be approximately

$$3n + \frac{n(n+1)}{2}$$

or

$$\frac{n^2}{2} + \frac{7}{2} n, \tag{113}$$

which does not include the preliminary calculations needed to obtain $x_1(t)$ and $\beta(t)$. Since $L = nh$, the estimate may also be expressed as

$$\frac{L^2}{2h^2} + \frac{7L}{2h}, \tag{114}$$

indicating that the errors (which vary as h^2) will be an inverse function of the effort expended.

This estimate compares rather unfavorably with other approximate methods for which the operations per step is a constant. In defense of the proposed method, it may be said (1) that the number of operations is independent of the order of the equation being solved and (2) that the use of the entire past history of one of the variables helps to make possible the accuracies attainable by this method. In special cases, if $\beta(t)$ approaches a constant for large t, some of this work may be avoided.

Conclusions

A method for the approximate determination of the response of a nonlinear system to fairly arbitrary inputs has been described. The method is presently applicable to systems with one essential nonlinearity whose mathematical relations can be indicated by a standard type of block diagram. The techniques involved -- operational mathematics, numerical integration, and graphical solution of algebraic equations -- are generally known to engineers, who will find the accuracy of the method satisfactory for most purposes.

It appears that the method is applicable to systems in which the nonlinearity changes in a prescribed manner with time or depends on an independent signal, such as a grid voltage of a vacuum tube. With some additional work, it may prove applicable to systems having more than one nonlinear element.

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